

AN APPROACH TO CLUSTER STRUCTURES ON MODULI OF LOCAL SYSTEMS FOR GENERAL GROUPS

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ABSTRACT. Let S be a surface, G a simply-connected classical group, and G' the associated adjoint form of the group. In [FG1], it was shown that the moduli spaces of framed local systems $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ have the structure of cluster varieties, and thus together form a cluster ensemble, when G had type A . This was extended to classical groups in [Le]. In this paper we give an algorithm for constructing the cluster structure for general reductive groups G . The algorithm can be carried out under some mild hypotheses, which we explain, and which we believe hold in general. We show that these hypotheses hold when G has type G_2 , and therefore we are able to construct the cluster structure in this case. We also illustrate our approach by rederiving the cluster structure for G of type A . Our goals are to give some heuristics for the approach taken in [Le], point out the difficulties that arise for more general groups, and to record some useful calculations. Forthcoming work by Goncharov and Shen gives a different approach to constructing the cluster structure on $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$. We hope that some of the ideas here complement their more comprehensive work.

CONTENTS

1. Introduction	2
2. Background	3
2.1. Setup	3
2.2. Definition of the spaces $\mathcal{X}_{G,S}$ and $\mathcal{A}_{G,S}$	4
2.3. Relation to configurations of flags	5
2.4. Reduction to the case of $\text{Conf}_m \mathcal{A}$ or $\text{Conf}_m \mathcal{B}$	6
2.5. Cluster algebras	6
3. General groups	8
3.1. The cluster algebra on B^-	8
3.2. Extending the cluster structure to $\text{Conf}_3 \mathcal{A}_G$	13
4. Dynkin automorphisms	19
5. G has type G_2	20
5.1. The functions	21
5.2. The first transposition	24
5.3. The second transposition	26
5.4. The third transposition, Langlands duality	28
5.5. The sequence of mutations for a flip	30
5.6. The space $\mathcal{X}_{G,S}$	37
References	39

1. INTRODUCTION

Let S be a topological surface S of with non-empty boundary, let G be a simply connected semi-simple group, and let G' be the adjoint form of this group. We are interested in the space $\mathcal{L}_{G,S}$, the moduli space of G -local systems on the surface S , or, equivalently, the space of representations of $\pi_1(S)$ into G . There are two closely related spaces $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$, which were constructed by Fock and Goncharov [FG1]. Both the the spaces $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ are variations on the space $\mathcal{L}_{G,S}$; they parameterize local systems with certain types of framing at the boundary of S .

One advantage of studying the spaces $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ is that they have rational co-ordinate charts. In many cases it is known that these rational co-ordinate charts come from the structure of these spaces as *cluster varieties*. In these cases, the pair of spaces $(\mathcal{A}_{SL_n,S}, \mathcal{X}_{PGL_n,S})$ form what Fock and Goncharov call a *cluster ensemble*. This was shown for G of type A in [FG1] and extended to types B, C, D in [Le]. In this paper we give an approach that we expect to work for all reductive groups, carrying out the approach in full when G has type G_2 , and also showing how it can be used to rederive the cluster structure in type A . (A more comprehensive approach can be found in upcoming work of Goncharov and Shen.)

The existence of these cluster structures on $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ means that both these spaces have an atlas of coordinate charts such that all transition functions involve only addition, multiplication and division. In other words, these spaces each have a *positive atlas* and may be called *positive varieties*. Thus these spaces exhibit the phenomenon of total positivity discovered by Lusztig [Lu]. These positive structures coincide with those discovered in [FG1], and can be used to derive results in higher Teichmuller theory.

In this paper, we give an algorithm for constructing cluster ensemble structures on $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ for any reductive group G . The success of this algorithm depends on showing that the functions we construct lie in a single tensor invariant space

$$[V_\lambda \otimes V_\mu \otimes V_\nu]^G.$$

Here V_λ is the irreducible representation of G with highest weight λ . In [Le], this computation was carried out for G of type B, C or D . In this paper, we carry out these computations in type A and G_2 , thus extending the results of [Le].

An important building block for the cluster structures on $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ comes from the cluster structures on double Bruhat cells and in flag varieties constructed in [BFZ], and based on earlier work found in [FZ], [BZ]. The cluster structures on $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ are constructed via ideal triangulations of S . The cluster structure on each triangle is close to the cluster structure on the Borel subgroup B for the group G . The first main problem is to correctly identify the cluster structure on each triangle and to understand how these structures glue together. We were inspired in our solution to this problem by the idea of “amalgamation” developed in [FG3].

For each triangulation of the surface S and for any ordering of the vertices in each triangle of the triangulation, one can write down an associated seed for the cluster ensemble. (This seed can be constructed out of a reduced word for the longest element w_0 of the Weyl group of G .) When $G = SL_n$ or PGL_n these functions (for a particular choice of the reduced word for w_0) somewhat miraculously exhibit S_3 symmetry. This is not the case for general groups G . Therefore, one would like to have sequences of mutations that relate the seeds coming from the various S_3 symmetries. Moreover, we one would like to relate seeds that correspond to different triangulations. This problem reduces to the problem of relating seeds coming from triangulations that differ by a “flip,” which is the change of triangulation that results from replacing one diagonal of a quadrilateral by another.

If one can show that S_3 symmetries and flips are realized by cluster mutation, one has an inclusion of the mapping class group of S into the cluster modular group of the cluster algebra on $\mathcal{A}_{G,S}$. When G has type A , the S_3 symmetry is automatic, while the sequence of mutations for a flip comes the octahedron recurrence. In [Le], we gave the appropriate sequences of mutations realizing S_3 symmetries and the flip when G was a classical group. In this paper, we give these sequences in the case where G has type G_2 . We would like to point out that this is one of the major remaining difficulties in types E and F , in addition to showing that the functions in the cluster algebra correspond to tensor invariants.

While writing this paper, we learned of forthcoming work of Goncharov and Shen that shows that $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ are cluster varieties for all reductive groups G . Nevertheless, we believe that our viewpoint, and some of the auxiliary results and calculations, are interesting in their own right.

Our hope is that in describing the procedure for constructing the cluster structure on $\mathcal{A}_{G,S}$ in a uniform way, we give some motivations for the constructions in [Le] that seemingly proceed on a case-by-case basis. Furthermore, the examples and computations in this paper can serve as an introduction to the more involved computations in [Le]. Let us collect here some of the goals of this paper:

- (1) To give a uniform way to construct the cluster algebra on $\mathcal{A}_{G,S}$ for reductive G .
- (2) To explain the heuristics we used in deriving the cluster structure for classical groups in [Le].
- (3) To carry out the algorithm for the construction of the cluster algebra in full in types A_n and G_2 .
- (4) To give one point of view on the elements of the cluster mapping class group coming from outer automorphisms of G .
- (5) To record some computations in types A_n and G_2 analogous to those performed in [Le] for types B, C, D .
- (6) To give a more conceptual derivation of the sequence of mutations relating the two reduced words for $w_0 \in G$ when G has type G_2 . This calculation was first performed in [BZ] and again in [FG3], and in both cases, the derivation was rather complicated.

Here is the outline of the paper. In Section 2, we review some of the work of Fock and Goncharov. We define the spaces $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ and relate them to spaces of configurations of points in G/B and G/U : $\text{Conf}_m \mathcal{A}_G$ and $\text{Conf}_m \mathcal{B}_G$. We also recall the necessary facts about cluster algebras.

In Section 3, we show how, starting with the constructions of [BFZ], one can derive the cluster structure on $\text{Conf}_3 \mathcal{A}_G$, assuming that the cluster functions lie in tensor invariant spaces. We also describe the data needed to glue triangles. Throughout this section, our running example will be type A . In Section 4, we explain how this construction allows one to realize the action of outer automorphisms of G on $\mathcal{A}_{G,S}$. In Section 5, we verify that the cluster functions lie in tensor invariant spaces in type G_2 , thus allowing us to construct the cluster algebra structure in this case. We also exhibit the sequences of mutations realizing S_3 symmetries and the flip.

2. BACKGROUND

2.1. Setup. Let S be a compact oriented surface, with or without boundary, and possibly with a finite number of marked points on each boundary component. We will refer to this whole set of data—the surface and the marked points on the boundary—by S . We will always take S to be hyperbolic, meaning it either has negative Euler characteristic, or contains enough marked points on the boundary (in other words, we can give it the structure of a hyperbolic surface

such that the boundary components that do not contain marked points are cusps, and all the marked points are also cusps).

Let G be a semi-simple algebraic group. When G is adjoint, i.e., has trivial center (for example, when $G = PGL_m$), we can define a higher Teichmüller space $\mathcal{X}_{G,S}$. On the other hand, for G simply-connected (for example, when $G = SL_m$), we can define the higher Teichmüller space $\mathcal{A}_{G,S}$. They will be the space of local systems of S with structure group G with some extra structure of a framing of the local system at the boundary components of S . Alternatively, these spaces describe homomorphisms of $\pi_1(S)$ into G modulo conjugation plus some extra data.

When S does have at least one hole, the spaces $\mathcal{X}_{G,S}$ and $\mathcal{A}_{G,S}$ have a distinguished collection of coordinate systems, equivariant under the action of the mapping class group of S . Using an elaboration of Lusztig's work on total positivity, one can show that all the transition functions between these coordinate systems are subtraction-free, and give a *positive atlas* on the corresponding moduli space. This positive atlas gives the spaces $\mathcal{X}_{G,S}$ and $\mathcal{A}_{G,S}$ the structure of a *positive variety*.

The existence of these extraordinary positive co-ordinate charts depends on Lusztig's theory of total positivity in semi-simple Lie groups [Lu] and is a reflection of the cluster algebra structure of the ring of functions on these spaces.

2.2. Definition of the spaces $\mathcal{X}_{G,S}$ and $\mathcal{A}_{G,S}$. The data of a framing of a local system involves the geometry of the flag variety associated to a group. Let B be a Borel subgroup, a maximal solvable subgroup of G . Then $\mathcal{B} = G/B$ is the flag variety. Let $U := [B, B]$ be a maximal unipotent subgroup in G . Then we will call $\mathcal{A} = G/U$ the “principal affine space” (sometimes also referred to as the “base affine space”). We will refer to elements of \mathcal{A} as “principal flags.”

Let \mathcal{L} be a G -local system on S . For any space X on equipped with a G -action, we can form the associated bundle \mathcal{L}_X . For $X = G/B$ we get the associated flag bundle $\mathcal{L}_{\mathcal{B}}$, and for $X = G/U$, we get the associated principal flag bundle $\mathcal{L}_{\mathcal{A}}$.

Definition 2.1. A framed G -local system on S is a pair (\mathcal{L}, β) , where \mathcal{L} is a G -local system on S , and β a flat section of the restriction of $\mathcal{L}_{\mathcal{B}}$ to the punctured boundary of S .

The space $\mathcal{X}_{G,S}$ is the moduli space of framed G -local systems on S .

The definition of the space $\mathcal{A}_{G,S}$ is slightly more complicated. It involves twisted local systems. We shall define this notion.

Let G be simply-connected. The maximal length element w_0 of the Weyl group of G has a natural lift to G , denoted \bar{w}_0 . Let $s_G := \bar{w}_0^2$. It turns out that s_G is in the center of G and that $s_G^2 = e$. Depending on G , s_G will have order one or order two. For example, for $G = SL_{2k}$, s_G has order two, while for $G = SL_{2k+1}$, s_G has order one.

The fundamental group $\pi_1(S)$ has a natural central extension by $\mathbb{Z}/2\mathbb{Z}$. We see this as follows. For a surface S , let $T'S$ be the tangent bundle with the zero-section removed. $\pi_1(T'S)$ is a central extension of $\pi_1(S)$ by \mathbb{Z} :

$$\mathbb{Z} \rightarrow \pi_1(T'S) \rightarrow \pi_1(S).$$

The quotient of $\pi_1(S)$ by the central subgroup $2\mathbb{Z} \subset \mathbb{Z}$, gives $\bar{\pi}_1(S)$ which is a central extension of $\pi_1(S)$ by $\mathbb{Z}/2\mathbb{Z}$:

$$\mathbb{Z}/2\mathbb{Z} \rightarrow \bar{\pi}_1(S) \rightarrow \pi_1(S).$$

Let $\sigma_S \in \bar{\pi}_1(S)$ denote the non-trivial element of the center.

A *twisted G -local system* is a representation $\bar{\pi}_1(S)$ in G such that σ_S maps to s_G . Such a representation gives a local system on $T'S$.

Now we must describe the framing data for a twisted local system. Let \mathcal{L} be a twisted G -local system on S . Such a twisted local system gives an associated principal affine bundle $\overline{\mathcal{L}}_{\mathcal{A}}$ on the punctured tangent bundle $T'S$. For any boundary component of S , we will construct sections of the punctured tangent bundle above these boundary components. Given any boundary component, consider the outward pointing unit tangent vectors along this component—this gives a section of the punctured tangent bundle above each boundary component of S . We get a bunch of loops and arcs in $T'S$ lie over the boundary of S . Call this the *lifted boundary*.

Definition 2.2. A *decorated G -local system* on S consists of (\mathcal{L}, α) , where \mathcal{L} is a twisted local system on S and α is a flat section of $\overline{\mathcal{L}}_{\mathcal{A}}$ restricted to the lifted boundary.

The space $\mathcal{A}_{G,S}$ is the moduli space of decorated G -local systems on S .

Note that in the case where $s_G = e$, a decorated local system is just a local system on S along with a flat section of $\mathcal{L}_{\mathcal{A}}$ restricted to the boundary. One can generally pretend that this is the case without much danger.

2.3. Relation to configurations of flags. The positive co-ordinate systems on $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ arise by rationally identifying them with spaces of configurations of flags. For more details, see [FG1].

Let S be a hyperbolic surface. Give it some hyperbolic structure such that the boundary components that do not contain marked points are cusps, and all the marked points are also cusps. The particular choice of hyperbolic structure will turn out not to matter. Then the universal cover of S will be a subset of the hyperbolic plane, and all these cusps will lie at the boundary at infinity of the hyperbolic plane. These cusps form a set C that has a cyclic ordering. C also carries a natural action of $\pi_1(S)$. The action of $\pi_1(S)$ preserves the cyclic ordering on C . The set C with its cyclic order is independent of our choice of hyperbolic structure on C . An *ideal triangulation* of S consists of a triangulation of S that has vertices at the cusps of S . An ideal triangulation of S induces an ideal triangulation of its universal cover. This triangulation of the universal cover will have its vertices at the set C .

A $\pi_1(S)$ -equivariant configuration of flags (respectively principal affine flags) parameterized by C is a map $\beta : C \rightarrow \mathcal{B}$ (respectively a map $\beta : C \rightarrow \mathcal{A}$) such that there is a map $\rho : \pi_1(S) \rightarrow G$ such that for $\gamma \in \pi_1(S)$,

$$\beta(\gamma \cdot c) = \rho(\gamma) \cdot c$$

for all points $c \in C$.

Starting with any point of $\mathcal{X}_{G',S}$ (respectively $\mathcal{A}_{G,S}$), we may look at the universal cover of S . On the universal cover, the local system becomes trivial, and the framing of the local system then gives a flag (respectively a principal affine flag) at each point of the cyclic set C . Thus any point in $\mathcal{X}_{G',S}$ (respectively $\mathcal{A}_{G,S}$) gives a $\pi_1(S)$ equivariant configuration of flags (respectively principal affine flags) parameterized by C .

Theorem 2.3. [FG1] The space $\mathcal{X}_{G',S}$ has a positive atlas that comes from identifying a framed local system with a $\pi_1(S)$ -equivariant positive configuration of flags parameterized by C .

The space $\mathcal{A}_{G,S}$ has a positive atlas that comes from identifying a decorated local system with a $\pi_1(S)$ -equivariant twisted positive cyclic configuration of principal affine flags parameterized by C .

This theorem gives a birational equivalence between $\mathcal{X}_{G',S}$ (respectively $\mathcal{A}_{G,S}$) and the stack of π_1 -equivariant positive configurations of flags (respectively principal affine flags) parameterized by C . This birational morphism is an isomorphism on positive points: the positive points

of $\mathcal{X}_{G',S}$ (respectively $\mathcal{A}_{G,S}$) correspond to positive representations of $\rho : \pi_1(S) \rightarrow G$ plus decoration. Any point of $\mathcal{X}_{G',S}(\mathbb{R}_{>0})$ (respectively $\mathcal{A}_{G,S}(\mathbb{R}_{>0})$) gives a $\pi_1(S)$ -equivariant positive configuration of flags parameterized by C . This configuration of flags uniquely determines the representation ρ as well as the decoration on the corresponding local system.

When S is a disk with marked points on the boundary we simply get moduli spaces of configurations of points in the flag variety $\mathcal{B} := G'/B$ and twisted configurations of points of the principal affine variety $\mathcal{A} := G/U$. For more details, see [FG1]

2.4. Reduction to the case of $\text{Conf}_m \mathcal{A}$ or $\text{Conf}_m \mathcal{B}$. We now recall the positive structures on the spaces $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ as constructed in [FG1].

In [FG1], the authors explain how to construct a positive co-ordinate chart on $\text{Conf}_m(\mathcal{B})$ for each triangulation of an m -gon. If we place the m flags at the vertices of an m -gon, then to each triangle in the triangulation of the m -gon, we get a configuration of three flags, and to this configuration of three flags, we attach some *face functions*. The face functions give a positive co-ordinate chart on $\text{Conf}_3(\mathcal{B})$. Any edge in the triangulation belongs to two triangles. We attach to each edge a set of *edge functions*, which depend on the four flags at the corners of the two triangles. For any edge, its edge functions along with the face functions for the triangles sharing that edge form a positive co-ordinate chart on $\text{Conf}_4(\mathcal{B})$. Thus the face functions give invariants of three flags, while the edge functions tell us how to glue two configurations of three flags into a configuration of four flags. Gluing along all edges of a triangulation will give a configuration of m flags.

There are also positive co-ordinate charts on $\text{Conf}_m(\mathcal{A})$ attached to each triangulation of an m -gon. These are constructed slightly differently. To each edge in the triangulation, we attach a set of *edge functions* which depend on the two flags at the ends of the edge. To each triangle in the triangulation, we get a configuration of three principal flags, and in addition to the edge functions of each pair of edges in the triangle, we attach some *face functions*, which depend on all three of the principal flags (not just two at a time). Thus whereas for $\text{Conf}_m(\mathcal{B})$, the edge functions give us the data for gluing configurations of three flags, for $\text{Conf}_m(\mathcal{A})$, two configurations of three principal flags can be glued along an edge only if the edge functions along that edge are identical. Thus we exchange gluing data for restrictions on when we can glue.

For a general surface S , we use the identification of $\mathcal{X}_{G',S}$ (respectively $\mathcal{A}_{G,S}$) with π_1 -equivariant configurations of flags (respectively, principal flags) parameterized by the cyclic set C . Any ideal triangulation of S gives a triangulation of the infinite polygon with vertices the points of the cyclic set C . We take the face and edge functions for this triangulation. Because of π_1 -equivariance, we get a finite set of functions coming from the edges and faces of the triangulation of S . This set of functions forms a positive co-ordinate chart for $\mathcal{X}_{G',S}$ (respectively $\mathcal{A}_{G,S}$). The co-ordinate charts coming from different triangulations of the surface give a positive atlas on $\mathcal{X}_{G',S}$ (respectively $\mathcal{A}_{G,S}$).

One of the goals of this paper will be to show that these edge and face functions can be realized as part of a cluster ensemble structure on the pair of spaces $(\mathcal{X}_{G',S}, \mathcal{A}_{G,S})$.

2.5. Cluster algebras. We review here the basic definitions of cluster algebras, following [W]. Cluster algebras are commutative rings that come equipped with a collection of distinguished sets of generators, called *cluster variables* or \mathcal{A} -coordinates. One can obtain one set of generators from another set of generators by a process called mutation.

Each set of generators belongs to a seed, which roughly consists of the set of generators along with a B -matrix. The B -matrix encodes how one mutates from one seed to any adjacent seed.

Starting from any initial seed, the process of mutation gives all the seeds (and all the sets of generators) for the cluster algebra. The cluster variables are coordinates on the \mathcal{A} -space.

The same combinatorial data underlying a seed gives rise to a second, related, algebraic structure, called \mathcal{X} -coordinates. The \mathcal{X} -coordinates are functions on the \mathcal{X} space. The \mathcal{A} -coordinates and \mathcal{X} -coordinates are related by a canonical monomial transformation, which gives a map from the \mathcal{A} -space to the \mathcal{X} -space. Together, the data of the \mathcal{A} -space and the \mathcal{X} -space, along with their distinguished sets of coordinates, is called a cluster ensemble.

Cluster algebras and \mathcal{X} -coordinates are defined by seeds. A seed $\Sigma = (I, I_0, B, d)$ consists of the following data:

- (1) An index set I with a subset $I_0 \subset I$ of “frozen” indices.
- (2) A rational $I \times I$ exchange matrix B . It should have the property that $b_{ij} \in \mathbb{Z}$ unless both i and j are frozen.
- (3) A set $d = \{d_i\}_{i \in I}$ of positive integers that skew-symmetrize B ; that is, $b_{ij}d_j = -b_{ji}d_i$ for all $i, j \in I$.

For most purposes, the values of d_i are only important up to simultaneous scaling. Also note that the values of b_{ij} where i and j are both frozen will play no role in the cluster algebra, though it is sometimes convenient to assign values to b_{ij} for bookkeeping purposes. These values become important in *amalgamation*, where one unfreezes some of the frozen variables.

Let $k \in I \setminus I_0$ be an unfrozen index of a seed Σ . We say another seed $\Sigma' = \mu_k(\Sigma)$ is obtained from Σ by mutation at k if we identify the index sets in such a way that the frozen variables and d_i are preserved, and the exchange matrix B' of Σ' satisfies

$$(1) \quad b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} & b_{ik}b_{kj} \leq 0 \\ b_{ij} + |b_{ik}|b_{kj} & b_{ik}b_{kj} > 0. \end{cases}$$

Two seeds Σ and Σ' are said to be mutation equivalent if they are related by a finite sequence of mutations.

To a seed Σ we associate a collection of *cluster variables* $\{A_i\}_{i \in I}$ and a split algebraic torus $\mathcal{A}_\Sigma := \text{Spec } \mathbb{Z}[A_I^{\pm 1}]$, where $\mathbb{Z}[A_I^{\pm 1}]$ denotes the ring of Laurent polynomials in the cluster variables. If Σ' is obtained from Σ by mutation at $k \in I \setminus I_0$, there is a birational *cluster transformation* $\mu_k : \mathcal{A}_\Sigma \rightarrow \mathcal{A}_{\Sigma'}$. This is defined by the *exchange relation*

$$(2) \quad \mu_k^*(A'_i) = \begin{cases} A_i & i \neq k \\ A_k^{-1} \left(\prod_{b_{kj} > 0} A_j^{b_{kj}} + \prod_{b_{kj} < 0} A_j^{-b_{kj}} \right) & i = k. \end{cases}$$

These transformations provide gluing data between any tori \mathcal{A}_Σ and $\mathcal{A}_{\Sigma'}$ of mutation equivalent seeds Σ and Σ' . The \mathcal{A} -space $\mathcal{A}_{|\Sigma|}$ is defined as the scheme obtained from gluing together all such tori of seeds mutation equivalent with an initial seed Σ .

Definition 2.4. Let Σ be a seed. The cluster algebra $\mathcal{A}(\Sigma)$ is the \mathbb{Z} -subalgebra of the function field of $\mathcal{A}_{|\Sigma|}$ generated by the collection of all cluster variables of seeds mutation equivalent to Σ . The upper cluster algebra $\overline{\mathcal{A}}(\Sigma)$ is

$$\overline{\mathcal{A}}(\Sigma) := \mathbb{Z}[\mathcal{A}_{|\Sigma|}] = \bigcap_{\Sigma' \sim \Sigma} \mathbb{Z}[\mathcal{A}_{\Sigma'}] \subset \mathbb{Q}(\mathcal{A}_{|\Sigma|}),$$

or the intersection of all Laurent polynomial rings in the cluster variables of seeds mutation equivalent to Σ .

Given a seed Σ we also associate a second algebraic torus $\mathcal{X}_\Sigma := \text{Spec } \mathbb{Z}[X_I^{\pm 1}]$, where $\mathbb{Z}[X_I^{\pm 1}]$ again denotes the Laurent polynomial ring in the variables $\{X_i\}_{i \in I}$. If Σ' is obtained from Σ by mutation at $k \in I \setminus I_0$, we again have a birational map $\mu_k : \mathcal{X}_\Sigma \rightarrow \mathcal{X}_{\Sigma'}$. It is defined by

$$(3) \quad \mu_k^*(X'_i) = \begin{cases} X_i X_k^{[b_{ik}]_+} (1 + X_k)^{-b_{ik}} & i \neq k \\ X_k^{-1} & i = k, \end{cases}$$

where $[b_{ik}]_+ := \max(0, b_{ik})$. The \mathcal{X} -space $\mathcal{X}_{[\Sigma]}$ is defined as the scheme obtained from gluing together all such tori of seeds mutation equivalent with an initial seed Σ .

Now we will describe the natural map from \mathcal{A}_Σ to \mathcal{X}_Σ . Let us assume that the entries of the B -matrix are all integers. Then we can define $p : \mathcal{A}_\Sigma \rightarrow \mathcal{X}_\Sigma$ by

$$p^*(X_i) = \prod_{j \in I} A_j^{B_{ij}}.$$

This formula appears to depend on the seed, but it actually intertwines the mutation of both the \mathcal{A} -coordinates and the \mathcal{X} -coordinates. In other words, if Σ' is obtained from Σ by mutation at k , there is a commutative diagram

$$\begin{array}{ccc} \mathcal{A}_\Sigma & \xrightarrow{\mu_k} & \mathcal{A}_{\Sigma'} \\ \downarrow p & & \downarrow p' \\ \mathcal{X}_\Sigma & \xrightarrow{\mu_k} & \mathcal{X}_{\Sigma'} \end{array}$$

3. GENERAL GROUPS

3.1. The cluster algebra on B^- . We describe here how one can find the cluster algebra structure on $\text{Conf}_n \mathcal{A}_G$ for any simply-connected simple group G . In particular, we believe this procedure will give the correct seeds for $\text{Conf}_n \mathcal{A}_G$ when G is of type E, F or G , although we will only carry out the process in full for type G_2 .

We first give an algorithm for constructing the cluster structure on $\text{Conf}_3 \mathcal{A}_G$. In [Le], for all classical groups G , we related the cluster structure on $\text{Conf}_3 \mathcal{A}_G$ to Berenstein, Fomin and Zelevinsky's cluster structure on B , the Borel in the group G ([BFZ]). Let us review how this story goes. Let (A_1, A_2, A_3) be a triple of principal flags in $\text{Conf}_3 \mathcal{A}_G$. There will be functions attached to these vertices that only depend on two of the three flags. We will call these the *edge functions*. Let us call all other functions, which depend on all three flags, *face functions*. We wish to construct the face functions, as well as the edge functions attached to the edges $A_1 A_2$, $A_2 A_3$, and $A_1 A_3$.

We first consider the subset of $\text{Conf}_3 \mathcal{A}_G$ given by triples of principal flags in the image of B^- under the map

$$i : b \in B^- \rightarrow (U^-, \overline{w_0} U^-, b \cdot \overline{w_0} U^-) \in \text{Conf}_3 \mathcal{A}_G.$$

The cluster structure on B^- will then a cluster structure on $\text{Conf}_3 \mathcal{A}_G$ with all the functions on the edge $A_1 A_2$ removed. This will give us all the face functions, as well as all the functions on the two edges $A_2 A_3$ and $A_1 A_3$. It also gives us all the arrows of the quiver connecting face vertices as well as face vertices with the vertices on edges $A_2 A_3$ and $A_1 A_3$.

Let us review how to construct the cluster structure on B^- . A more detailed account can be found in [BFZ] and [FG3].

Let w_0 be the longest word in the Weyl group of G . The double Bruhat cell

$$G^{w_0, e} := B^+ w_0 B^+ \cap B^- e B^-$$

is the open part of B^- . Take any reduced-word $s_{i_1} \dots s_{i_K}$ for w_0 . Our convention will be that we read the word from right to left, i.e., the simple reflection s_K followed by the simple reflection s_{K-1} , etc. Here $1 \leq i_j \leq n$, where n is the number of nodes in the Dynkin diagram. For each node i of the Dynkin diagram, there is a corresponding simple reflection s_i in the Weyl group W .

For each reduced-word expression for w_0 there is a corresponding seed for the cluster algebra on B^- . The B -matrix for this seed is encoded via a quiver which consists of $n + K$ vertices. Of these vertices, $2n$ are frozen, and these frozen vertices will correspond to edge vertices. Each vertex of the quiver belongs to one of the nodes of the Dynkin diagram. If the simple reflection s_i occurs a_i times in the reduced-word for w_0 , there will be $a_i + 1$ vertices belonging to the node i . Of these, two (the first and the last) will be frozen.

The multipliers d_i for the vertices are determined by the nodes they are associated with. If a node in the Dynkin diagram is associated with a short root, the multipliers for the vertices belong to this node are 1. If a node in the Dynkin diagram is associated with a long root, the multipliers for the vertices belong to this node are 2 in types B, C, F and 3 in type G . Alternatively, let the node i correspond to the root α_i , and let us normalize the lengths of the roots so that the short roots have length 1. Then the multipliers for the vertices belonging to node i are $(\text{length of } \alpha_i)^2$.

We have described the vertices of the quiver and the multipliers for these vertices. The arrows in the quiver carry the following information:

- An arrow from j to i means that $b_{ij} > 0$ and $b_{ji} < 0$.
- $|b_{ij}| = 2$ if $d_i = 2$ and $d_j = 1$.
- $|b_{ij}| = 3$ if $d_i = 3$ and $d_j = 1$.
- $|b_{ij}| = 1$ otherwise.

The procedure for constructing this quiver from the reduced word for w_0 is complicated, and perhaps unilluminating, to state in full generality. Instead, let us look at the example of SL_4 in detail.

One reduced word for $w_0 \in W_{SL_4}$ is

$$s_1 s_2 s_1 s_3 s_2 s_1.$$

The corresponding quiver is for B^- is

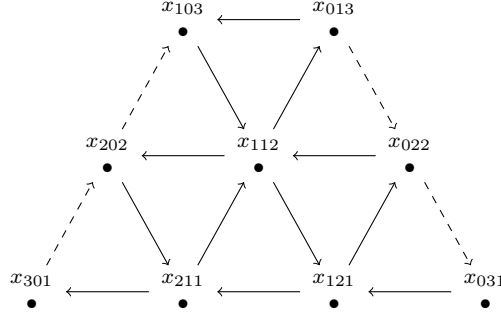
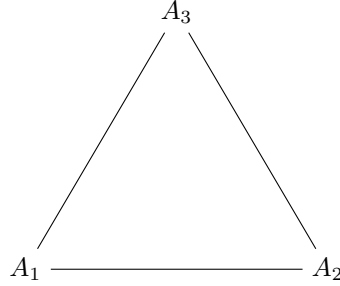


Figure 1. The quiver for the cluster algebra on $B_{SL_4}^-$.

For the moment, let us ignore the names of the vertices. In the above quiver, there are three rows. The vertices belonging to node 1 of the Dynkin diagram are on the bottom row; the vertices belonging to node 2 are in the middle row; and the vertices belonging to node 3 are on the top row. SL_n is a simply-laced group, so $d_i = 1$ for all vertices. The vertices on either end of each row are frozen vertices. The leftmost vertices in each row are the edge functions for the edge A_1A_3 . The rightmost vertices in each row are the edge functions for the edge A_2A_3 . The flags A_1, A_2, A_3 are oriented as follows:



The dotted arrows only go between frozen vertices. A dotted arrow between vertices i and j means that b_{ij} is half what it would be if the arrow were solid. Thus dotted arrows are “half-arrows.”

Each occurrence of the reflections s_1, s_2, s_3 gives a portion of the quiver as depicted in Figures 2-4:

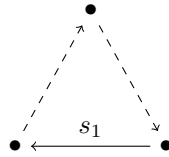


Figure 2. Portion of the quiver corresponding to the simple reflection s_1 .

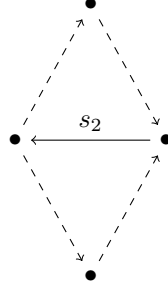


Figure 3. Portion of the quiver corresponding to the simple reflection s_2 .

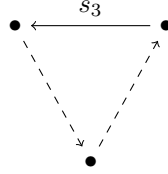


Figure 4. Portion of the quiver corresponding to the simple reflection s_3 .

We glue these pieces together according to the reduced word for w_0 as in Figure 5 to obtain the quiver for the cluster algebra. The triangles $x_{130}x_{220}x_{121}$, $x_{121}x_{211}x_{112}$, and $x_{112}x_{202}x_{103}$ correspond to the simple reflection s_1 . The quadrilaterals $x_{220}x_{121}x_{211}x_{310}$ and $x_{211}x_{112}x_{202}x_{301}$ correspond to the simple reflection s_2 . The triangles $x_{220}x_{310}x_{211}$ and $x_{211}x_{301}x_{202}$ correspond to the simple reflection s_3 .

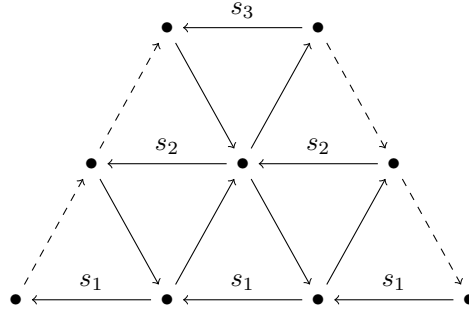


Figure 5. How the quiver corresponds to the reduced word $s_1s_2s_1s_3s_2s_1$ for $B_{SL_4}^-$.

The dotted arrows should be viewed as half a solid arrow. When we perform the gluing, two dotted arrows in the same direction glue to give us a solid arrow, whereas two dotted arrows in the opposite direction cancel to give us no arrow. (In the SL_4 example above, it turns out we never glue dotted arrows going in the opposite direction, but this does happen in general.) This

is the process of amalgamation. For general groups, the picture is similar: for a reduced word for w_0 , each simple reflection gives a portion of the quiver, and amalgamating these portions together gives the full quiver for the cluster algebra on B^- .

Let us now describe the functions attached to the vertices of the quiver. The functions are given by *generalized minors* of B^- , which we now define. Let $G_0 = U^- H U^+ \subset G$ be the open subset of elements of G having Gaussian decomposition $x = [x]_- [x]_0 [x]_+$. Then for any two elements $u, v \in W$, and any fundamental weight ω_i , we have the *generalized minor* $\Delta_{u\omega_i, v\omega_i}(x)$ defined by

$$\Delta_{u\omega_i, v\omega_i}(x) := ([\bar{u}^{-1}x\bar{v}]_0)^{\omega_i}.$$

The formula gives a well-defined value when $\bar{u}^{-1}x\bar{v} \in G_0$, but may have poles elsewhere.

We start with a reduced word for w_0 :

$$w_0 = s_{i_1} s_{i_2} s_{i_3} \cdots s_{i_{K-1}} s_{i_K}$$

For $1 \leq l \leq K$, we have the subword

$$u_l := s_{i_1} \cdots s_{i_l}.$$

In our situation, we are interested in the generalized minors $\Delta_{u\omega_i, v\omega_i}(x)$ when $u, v = e$, or when $v = e$ and $u = u_l = s_{i_1} s_{i_2} s_{i_3} \cdots s_{i_l}$ for $1 \leq l \leq K$.

Then the cluster functions on B^- given in [BFZ] are as follows:

- For each i , $1 \leq i \leq n$, we have the functions $\Delta_{\omega_i, \omega_i}$ (these are the n functions associated to $u, v = e$).
- For each l , $1 \leq l \leq K$, we have the functions

$$\Delta_{u_l \omega_{i_l}, \omega_{i_l}}$$

associated to $v = e$ and $u = u_l$.

Let us now match the functions with the vertices of the quiver. Suppose that in the reduced word for w_0 that we have chosen, there are a_i occurrences of the simple reflection s_i . Then the quiver contains $a_i + 1$ vertices belonging to the node i of the Dynkin diagram. Let us call these vertices $x_{i0}, x_{i1}, x_{i2}, \dots, x_{ia_i}$. (This labelling does not coincide with the labelling of the vertices in our example; the logic behind the labelling in our example will become clear later. This labelling does, however agree with the conventions used in [Le].)

Then the function $\Delta_{\omega_i, \omega_i}$ will be attached to the vertex x_{i0} . If s_{i_l} is the j^{th} occurrence of the simple reflection s_i in the reduced word for w_0 , then the function $\Delta_{u_l \omega_{i_l}, \omega_{i_l}}$ will be attached to the vertex x_{ij} .

The vertices along the edge $A_1 A_3$ will be x_{i0} for $1 \leq i \leq n$, and the associated functions are $\Delta_{\omega_i, \omega_i}$. The vertices along the edge $A_2 A_3$ will be x_{ia_i} and the associated functions are also edge functions. All the remaining functions are face functions.

Let us compute these functions in our example where $G = SL_4$. Straightforward computation gives that the functions $\Delta_{u\omega_i, v\omega_i}$ that we are interested in are given by minors with the following rows and columns:

Function	columns	rows
$\Delta_{\omega_1, \omega_1}$	1	1
$\Delta_{s_1 \omega_1, \omega_1}$	1	2
$\Delta_{s_1 s_2 s_1 \omega_1, \omega_1}$	1	3
$\Delta_{w_0 \omega_1, \omega_1}$	1	4
$\Delta_{\omega_2, \omega_2}$	1, 2	1, 2
$\Delta_{s_1 s_2 \omega_2, \omega_2}$	1, 2	2, 3
$\Delta_{s_1 s_2 s_1 s_3 s_2 \omega_2, \omega_2}$	1, 2	3, 4
$\Delta_{\omega_3, \omega_3}$	1, 2, 3	1, 2, 3
$\Delta_{s_1 s_2 s_1 s_3 \omega_3, \omega_3}$	1, 2, 3	2, 3, 4

More generally, when $G = SL_n$, the Weyl group is generated by simple reflections s_1, s_2, \dots, s_{n-1} . The reduced word we will use is

$$w_0 = s_1 s_2 \dots s_{n-1} s_1 s_2 \dots s_{n-2} \dots s_1 s_2 s_3 s_1 s_2 s_1.$$

In this word, there are $n - k$ occurrences of the simple reflection s_k . The functions $\Delta_{\omega_i, \omega_i}$ will be given by minors consisting of the first i columns and the first i rows. Suppose that s_{i_l} is the j^{th} occurrence of the simple reflection s_i in the reduced word for w_0 . Then the function $\Delta_{u_l \omega_i, \omega_i}$ will be given by the minor consisting of the first i columns and rows $j + 1, j + 2, \dots, j + i$.

3.2. Extending the cluster structure to $\text{Conf}_3 \mathcal{A}_G$. The construction of the previous section determines the cluster structure on the triangle $A_1 A_2 A_3$ with the edge $A_1 A_2$ removed. In order to complete the construction we need to do the following:

- (1) Construct the functions attached to the edge $A_1 A_2$.
- (2) Determine the multipliers for the functions attached to the edge $A_1 A_2$.
- (3) Determine the arrows in the quiver connecting the functions on the edge $A_1 A_2$ to the face functions.
- (4) Determine the (possibly fractional) arrows in the quiver connecting the functions on the edge $A_1 A_2$ to the other edge functions. item Determine the (possibly fractional) arrows in the quiver connecting the functions on the edge $A_1 A_2$ to each other.

Let us explain how to carry out these steps in turn. First, a simple calculation shows that the functions along the edges $A_1 A_3$ and $A_2 A_3$ as determined above are given by the canonical invariants of the tensor product

$$[V_\lambda \otimes V_{\lambda^\vee}]^G.$$

where λ runs over the fundamental weights ω_i , and λ^\vee is the weight corresponding to the representation dual to V_λ , in other words, $\lambda^\vee = -w_0(\lambda)$. It is then natural to construct the functions along the edge $A_1 A_2$ similarly. This determines all the functions for the cluster algebra on $\text{Conf}_3 \mathcal{A}$.

Second, the multipliers for all the functions except for those on the edge $A_1 A_2$ are determined by the construction of [BFZ]. One only needs to observe that the multiplier attached to the edge function which is given by the invariant in

$$[V_{\omega_i} \otimes V_{\omega_i^\vee}]^G$$

is given by the multiplier attached to the i -th node of the Dynkin diagram.

The third step is somewhat more subtle. We now know all the functions for the cluster algebra, and just need to determine the arrows in the quiver. The algebra of functions on $\text{Conf}_3 \mathcal{A}_G$ is naturally graded by dominant weights of G . In order to carry out our algorithm for determining the arrows, we need the following conjecture to hold:

Conjecture 3.1. Each of the functions $\Delta_{u_l \omega_{i_l}, \omega_{i_l}}$ lies in a single graded piece

$$[V_\lambda \otimes V_\mu \otimes V_\nu]^G$$

of $\mathcal{O}(\text{Conf}_3 \mathcal{A}_G)$, for some dominant weights λ, μ, ν .

There are good reasons to believe this conjecture, among them the Duality Conjectures of Fock and Goncharov, [FG1]. We also know that the conjecture holds in many cases.

Proposition 3.2. The conjecture holds in type A .

Proof. Recall that for SL_n , the function $\Delta_{u_l \omega_i, \omega_i}$ was given by the minor consisting of the first i columns and rows $j+1, j+2, \dots, j+i$. This means that it is given by an invariant in the space

$$[V_{\omega_k} \otimes V_{\omega_j} \otimes V_{\omega_i}]^G$$

where $k = n - i - j$. If U^- is given by the flag consisting of the forms

$$A_{1k} := (-1)^k e_{n-k+1} \wedge e_{n-k+2} \wedge \dots \wedge e_n,$$

and $\overline{w_0}U^-$ is given by the flag consisting of the forms

$$A_{2j} := e_1 \wedge e_2 \wedge \dots \wedge e_j,$$

and the columns of $b \in B^-$ are given by vectors v_1, \dots, v_n , so that $b\overline{w_0}U^-$ is given by forms

$$A_{3i} := v_1 \wedge v_2 \wedge \dots \wedge v_i,$$

then $\Delta_{u_l \omega_i, \omega_i}$ is given by

$$A_{1k} \wedge A_{2j} \wedge A_{3i}.$$

In our example of SL_4 , we get that the invariant attached to the vertex x_{ijk} lies in

$$[V_{\omega_i} \otimes V_{\omega_j} \otimes V_{\omega_k}]^G.$$

□

Remark 3.3. Although we can calculate that $\Delta_{u_l \omega_i, \omega_i}$ is given by a tensor product invariant in many cases, we don't know of any reason why this is the case. It would be interesting to find a more satisfying explanation of why generalized minors are given by tensor product invariants.

Theorem 3.4. The conjecture holds in types B, C, D .

The above theorem was proven in [Le], where we verified that the functions $\Delta_{u_l \omega_i, \omega_i}$ were given by tensor product invariants. In fact, we identified these invariants using webs. Thus our conjecture is true for all the classical groups. We show later that it holds in type G_2 , and we expect this to hold in general. In the following constructions, we assume that this is the case.

In order to complete out our construction of the arrows between vertices on the edge $A_1 A_2$ and face vertices, we need to use the following heuristic: we expect that the space $\text{Conf}_3 \mathcal{A}_G$ has as its corresponding \mathcal{X} -variety the space $\text{Conf}_3 \mathcal{B}_G$. Thus, for all the unfrozen vertices (which correspond to face functions), we need that the corresponding X -coordinate is a function on $\text{Conf}_3 \mathcal{B}_G$. A rational function on $\text{Conf}_3 \mathcal{A}_G$ descends to $\text{Conf}_3 \mathcal{B}_G$ if and only if it is in the $(0, 0, 0)$ -th graded piece of $\mathcal{O}(\text{Conf}_3 \mathcal{A}_G)$. Because we have that

$$p^*(X_i) = \prod_{j \in I} A_j^{B_{ij}},$$

if A_j lies in the graded piece $(\lambda_j, \mu_j, \nu_j)$, we need

$$(4) \quad \sum_{j \in I} B_{ij}(\lambda_j, \mu_j, \nu_j) = 0$$

for all $i \in I$ which is unfrozen. It is not difficult to see that condition 4 determines uniquely the arrows between the face vertices and the vertices on edge A_1A_2 . The existence of a choice of the values B_{ij} for these arrows satisfying this condition can be checked on a case-by-case basis. This completes the third step.

Fourth, we need to deal with the (possibly fractional) arrows between vertices on A_1A_2 and other edges. There are no \mathcal{X} -coordinates attached to the edge vertices. However, we expect that when we glue two triangles together, the edge vertices will then have \mathcal{X} -coordinates attached.

Suppose we have a quadrilateral $ABCD$, which is glued from triangles ABC and ACD along edge AC :

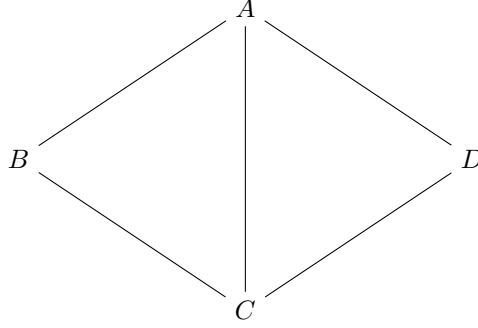


Figure 6. Quadrilateral $ABCD$.

$$[V_{\omega_i} \otimes V_{\omega_i^\vee}]^G$$

Let us consider an edge vertex i on the edge AC . Let us suppose that the function attached to vertex i lies in the invariant space

$$[V_\lambda \otimes V_{\lambda^\vee}]^G,$$

where the grading with respect to the flag at C is λ and the grading with respect to the flag at A is $\lambda^\vee := -w_0(\lambda)$. Here λ will be a fundamental weight for G . Suppose that $\lambda = \omega_k$. Each function A_j in the triangle ABC lies in some graded piece $(\lambda_j, \mu_j, \nu_j, 0)$ of $\mathcal{O}(\text{Conf}_4 \mathcal{A}_G)$. Then in the triangle ABC , we can again form the sum

$$S_{CA,B,\omega_k} = \sum_{j \in I} B_{ij}(\lambda_j, \mu_j, \nu_j, 0)$$

where we allow fractional values of B_{ij} for arrows between frozen vertices. Now, we can consider the triangle ACD . If from the point of view of triangle ABC , vertex i is associated with the fundamental weight ω_k , then from the point of view of triangle ACD , the vertex i is associated with the fundamental weight ω_k^\vee . Each function A_j in the triangle ACD lies in some graded piece $(\lambda_j, 0, \mu_j, \nu_j)$ of $\mathcal{O}(\text{Conf}_4 \mathcal{A}_G)$. Then in the triangle ACD , we can again form the sum

$$S_{AC,D,\omega_k^\vee} = \sum_{j \in I} B_{ij}(\lambda_j, 0, \mu_j, \nu_j).$$

Then in order for the vertex i to have an associated \mathcal{X} -coordinate, we require that

$$(5) \quad S_{CA,B,\omega_k} + S_{AC,D,\omega_k^\vee} = 0.$$

This has several implications. First, this means that

$$(6) \quad S_{AC,D,\omega_k^\vee} = (\lambda, 0, \mu, 0)$$

for some weights λ and μ . This is enough to determine the arrows between vertices on the edge A_1A_2 and other edge vertices. This completes the fourth step.

Moreover, we also expect that as we mutate the face vertices in $\text{Conf}_3 \mathcal{A}_G$, the value of S_{CA,B,ω_k} should not change. This is because mutating face vertices in the triangle ABC leaves the functions in triangle ACD untouched, and for equation 5 to hold, S_{CA,B,ω_k} must remain constant. Thus we expect that λ and μ are completely determined by the weight ω_k . In other words, there exist λ_k and μ_k such that in any triangle ABC , S_{AB,C,ω_k} has weight λ_k at A and weight μ_k at B . Denote $\omega_{k*} := \omega_k^\vee$. Then the equation $S_{CA,B,\omega_k} + S_{AC,D,\omega_{k*}} = 0$ means that

$$(7) \quad \lambda_k + \mu_{k*} = 0 = \lambda_{k*} + \mu_k$$

We know from the previous steps of our construction the values for $S_{A_3A_1,A_2,\omega_k}$ and $S_{A_2A_3,A_1,\omega_k}$. If we are to be able to the cluster structure on $\text{Conf}_3 \mathcal{A}_G$ to get cluster structures on $\text{Conf}_m \mathcal{A}_G$, we need $S_{A_3A_1,A_2,\omega_k} = S_{A_2A_3,A_1,\omega_k}$. Our procedure above through step 4 the determines values for $S_{A_3A_1,A_2,\omega_k}$ and $S_{A_2A_3,A_1,\omega_k}$. We then have:

Conjecture 3.5. Then $S_{A_3A_1,A_2,\omega_k} = S_{A_2A_3,A_1,\omega_k}$.

Provided these are equal, we then know the values for $S_{A_1A_2,A_3,\omega_k}$. This determines the arrows between the vertices on the edge A_1A_2 and completes the final step. The resulting cluster structure can then be glued to get a cluster structure on $\text{Conf}_m \mathcal{A}$ for any integer m . Note that our construction was conditional upon only Conjecture 3.1 and Conjecture 3.5. Once these conjectures hold, the procedure determines the cluster structure uniquely.

Let us illustrate this in an example. For SL_4 , the above weight analysis uniquely determines the following quiver:

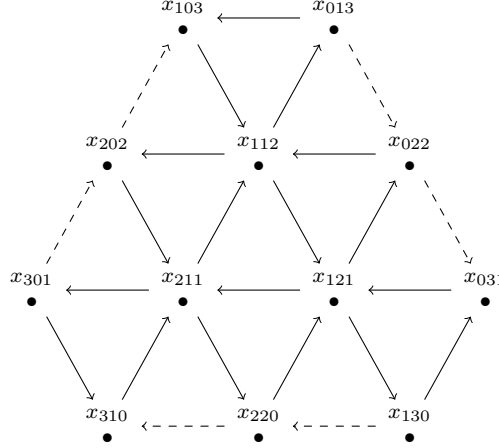


Figure 7. The quiver for the cluster algebra on $\text{Conf}_3 \mathcal{A}_{SL_4}$.

The arrows in and out of vertices x_{211} and x_{121} are forced by equation 4. The arrows from x_{301} to x_{310} and from x_{130} to x_{031} are determined by equation 6. One can check then that for the vertices on edges A_1A_3 and A_2A_3 , equation 7 holds. Finally, the dotted arrows on the bottom row are determined by the fact that λ_k and μ_k only depend on ω_k .

The function at x_{301} has weight ω_3 at A_1 and weight ω_1 at A_3 . We can then calculate, for example, that

$$S_{A_3A_1,A_2,\omega_1} = \left(-\omega_3 + \frac{\omega_2}{2}, 0, \omega_1 - \frac{\omega_2}{2}\right),$$

$$S_{A_3 A_1, A_2, \omega_2} = (-\omega_2 + \frac{\omega_1 + \omega_3}{2}, 0, \omega_2 - \frac{\omega_1 + \omega_3}{2}),$$

$$S_{A_3 A_1, A_2, \omega_3} = (-\omega_1 + \frac{\omega_2}{2}, 0, \omega_3 - \frac{\omega_2}{2}).$$

The computation above suggests the following conjecture:

Conjecture 3.6. S_{AB, C, ω_k} has weight $\frac{\alpha_k}{2}$ at A and weight $\frac{w_0(\alpha_k)}{2}$ at B . In other words, $\lambda_k = \frac{\alpha_k}{2}$ and $\mu_k = \frac{w_0(\alpha_k)}{2}$.

This conjecture goes beyond what we need to construct the cluster structure on $\text{Conf}_3 \mathcal{A}_G$. The conjecture automatically implies that equation 7 holds, for it implies that $\lambda_k = \frac{\alpha_k}{2}$ and $\mu_{k^*} = \frac{w_0(\alpha_{k^*})}{2} = -\frac{\alpha_k}{2}$.

Proposition 3.7. The conjecture holds for classical groups.

Proof. This can be done by explicit calculation in type A following from the calculations in the proof of Proposition 3.2. For other classical groups, this was verified in [Le]. \square

Thus, we have a procedure for constructing the cluster structure on $\text{Conf}_3 \mathcal{A}_G$ from any reduced word for w_0 , along with the fractional arrows we need to glue triangles together. In fact, our construction leads to six different cluster structures on $\text{Conf}_3 \mathcal{A}_G$. Three of these cluster structures just come from taking cyclic permutations cluster we constructed. In other words, we can apply the above procedure to the triangles $A_2 A_3 A_1$ and $A_3 A_1 A_2$ rather than $A_1 A_2 A_3$.

More precisely, let $\sigma_{(123)} \in S_3$ be an even permutation. $\sigma_{(123)}$ naturally acts on $\text{Conf}_3 \mathcal{A}_G$. Then consider the cluster structure coming from taking the same quiver as before, except that if the function f_i was originally attached to vertex i , now attach the function $\sigma_{(123)}^* f_i$ to the vertex i . This will give the cluster coming from the symmetry $\sigma_{(123)}$.

There is also a cluster structure on $\text{Conf}_3 \mathcal{A}_G$ coming from any odd permutation in S_3 . Let us consider the transposition $\sigma_{12} \in S_3$. To obtain this cluster structure coming from this transposition, take the original quiver and reverse all the arrows. Then if the function f_i was originally attached to vertex i , now attach the function $\pm \sigma_{(12)}^* f_i$ to the vertex i . The correct sign to take here is subtle.

Here is a more precise way to construct this cluster structure which pins down this sign. Suppose that we have a cluster constructed on the triangle $A_1 A_2 A_3$ using our above construction for the word

$$w_0 = s_{i_1} s_{i_2} \dots s_{i_K-1} s_{i_K}.$$

Then take the word

$$w_0 = s_{i_K} s_{i_K-1} \dots s_{i_2} s_{i_1}.$$

The cluster corresponding to this new word is the same as the result of applying the reflection interchanging A_1 and A_3 to the original cluster and reversing all arrows.

The following theorem is due to Berenstein, Fomin and Zelevinsky [BFZ]:

Theorem 3.8. Given any two reduced word expressions for w_0 , the resulting cluster structures on B^- are related by a sequence of cluster mutations.

Proposition 3.9. Suppose that we can construct a cluster structure on $\text{Conf}_3 \mathcal{A}$ using a reduced word for w_0 as above. Then we can construct a cluster structure on $\text{Conf}_3 \mathcal{A}$ using any reduced word for w_0 .

Proof. The reduced word for w_0 determines the face functions and the functions on edges A_1A_3 and A_2A_3 . It also determines the (fractional) arrows except those involving a vertex on A_1A_2 . Note that steps 1-5 in the above procedure above determined the cluster structure on $\text{Conf}_3 \mathcal{A}$ uniquely. Suppose we have two reduced words $s_{i_1}s_{i_2}\dots s_{i_K-1}s_{i_K}$ and $s_{j_1}s_{j_2}\dots s_{j_K-1}s_{j_K}$, and that the reduced word $s_{i_1}s_{i_2}\dots s_{i_K-1}s_{i_K}$ gives a cluster structure on $\text{Conf}_3 \mathcal{A}$. Then [BFZ] gives a sequence of mutations relating the cluster structure on B^- corresponding to $s_{i_1}s_{i_2}\dots s_{i_K-1}s_{i_K}$ to the cluster structure corresponding to $s_{j_1}s_{j_2}\dots s_{j_K-1}s_{j_K}$. Let us apply this same sequence of mutations to our cluster structure on $\text{Conf}_3 \mathcal{A}$.

We claim that this gives the cluster structure on $\text{Conf}_3 \mathcal{A}$ corresponding to $s_{j_1}s_{j_2}\dots s_{j_K-1}s_{j_K}$. The sequence of mutations only involves face vertices. Therefore throughout the sequence of mutations, the equations 4 continue to hold, and the values of $S_{A_3A_1,A_2,\omega_k}$, $S_{A_2A_3,A_1,\omega_k}$ and $S_{A_1A_2,A_3,\omega_k}$ remain constant. By the uniqueness of our construction, we end up with the cluster structure corresponding to the reduced word $s_{j_1}s_{j_2}\dots s_{j_K-1}s_{j_K}$. \square

Corollary 3.10. In the situations where we have a cluster algebra structure on $\text{Conf}_3 \mathcal{A}$ as constructed above, the cluster coming from the transposition interchanging A_1 and A_3 can be related to the original cluster by a sequence of mutations.

The cluster structure for $\text{Conf}_m \mathcal{A}$ comes from triangulating an m -gon and then attaching the cluster structure on $\text{Conf}_3 \mathcal{A}$ to each triangle. Let us make this more precise. On $\text{Conf}_3 \mathcal{A}_G$, the edge vertices are frozen vertices. Attached to these vertices are functions which are invariants of tensor products $[V_{\omega_i} \otimes V_{\omega_i^\vee}]^G$. Because edge vertices are frozen, this is true in all the clusters for $\text{Conf}_3 \mathcal{A}$.

To form the quiver for $\text{Conf}_m \mathcal{A}_G$, we first take a triangulation of an m -gon. On each of the $m-2$ triangles, attach any one of the six quivers formed from performing S_3 symmetries on the quiver for $\text{Conf}_3 \mathcal{A}_G$ described above. Each edge of each of these triangles has n frozen vertices. Let us describe how to glue two triangles together.

Let T_1, T_2 be two triangles with edges e_1, e_2, e_3 and e_4, e_5, e_6 , respectively. Suppose that we would like to glue the edges e_1 and e_4 . e_1 and e_4 each have n frozen vertices. We will glue these $2n$ vertices together in pairs to form n vertices. Each frozen vertex is glued to another vertex that shares the same function. These vertices then become unfrozen. If vertices i and j are glued with i' and j' to get new vertices i'' and j'' , then we declare that

$$b_{i''j''} = b_{ij} + b_{i'j'}.$$

In other words, two dotted arrows in the same direction glue to give us a solid arrow, whereas two dotted arrows in the opposite direction cancel to give us no arrow. One can easily check that any gluing will result in no dotted arrows using the unfrozen vertices. The arrows involving vertices that were not previously frozen remain the same.

Let us note here, that though for any reduced word for w_0 , we have six clusters that we can construct on $\text{Conf}_3 \mathcal{A}$, we do not know how to realize the S_3 symmetries via a sequence of mutations relating these six clusters in general. Moreover, one wishes to have a sequence of mutations realizing any change in triangulation. This comes down to finding a sequence of mutations realizing a flip—a change in a quadrilateral from the triangulation using one diagonal to the triangulation using the other diagonal.

In type A , the quiver for the reduced word

$$w_0 = s_1s_2\dots s_{n-1}s_1s_2\dots s_{n-2}\dots s_1s_2s_3s_1s_2s_1$$

is magically S_3 -symmetric, and the flip is realized by the octahedron recurrence. In types B, C, D , the realization of S_3 symmetries and flips via sequences of mutations made up the

bulk of the computations in [Le]. We will later carry out these computations in type G_2 . We have the following conjecture, that we know is true in types A, B, C, D, G :

Conjecture 3.11. For any reductive group G , the six clusters coming from applying S_3 symmetries to the original cluster structure on $\text{Conf}_3 \mathcal{A}$ can be realized via a sequence of mutations. Moreover, on $\text{Conf}_4 \mathcal{A}$, the clusters coming from the two different triangulations can be related by a sequence of mutations. Thus we have a cluster on $\text{Conf}_m \mathcal{A}$ attached to any choice of the following data: a triangulation of the m -gon, an ordering of the vertices in each triangle, and a choice of a reduced word for w_0 on each triangle. All of these clusters are related by sequences of mutations, and hence belong to the same cluster algebra.

4. DYNKIN AUTOMORPHISMS

As an application of the construction of the previous section, let us show how to realize the action of outer automorphisms of the group G on $\mathcal{A}_{G,S}$.

Let $\text{Aut}(G)$ denote the group of automorphisms of the group G . G acts on itself by conjugation, so it gives rise to an action of the group of inner automorphisms $\text{Inn}(G)$. $\text{Inn}(G)$ is normal inside $\text{Aut}(G)$. Then let $\text{Out}(G) := \text{Aut}(G)/\text{Inn}(G)$ denote the group of outer automorphisms of G . It is well known that $\text{Out}(G)$ is a finite group given by automorphisms of the Dynkin diagram of G .

We can realize the action of $\text{Out}(G)$ on G in the following way: $\text{Out}(G)$ acts on the Dynkin diagram, so it permutes the simple roots α . This induces a natural action of $\text{Out}(G)$ on the Cartan subgroup H . It also induces an action of $\text{Out}(G)$ on the generators x_α, y_α . This gives an action of $\text{Out}(G)$ on G , which is generated by H, x_α, y_α .

The above discussion means that $\text{Out}(G)$ naturally acts on the flag varieties G/U and G/B . It also acts on local systems in the following way. Let $\rho : \pi_1(S) \rightarrow G$ be a representation. An element $\sigma \in \text{Out}(G)$ acts on the set of such representations by post-composition:

$$\rho : \pi_1(S) \rightarrow G \xrightarrow{\sigma} G.$$

These actions of $\text{Out}(G)$ on flag varieties and local systems are compatible, and hence give rise to actions on $\mathcal{A}_{G,S}$ and $\mathcal{X}_{G,S}$. Let $\sigma \in \text{Out}(G)$. Then pulling back functions via σ gives an action of σ on $\mathcal{O}(\mathcal{A}_{G,S})$. We are interested in whether the action of σ preserves the cluster structure. In other words, acting by σ gives a potentially different cluster structure on $\mathcal{A}_{G,S}$. We would like to show that we in fact get the same cluster structure.

In order to show that σ preserves the cluster structure on $\mathcal{A}_{G,S}$, it is enough to show the following: there is some seed (I, I_0, B, d) with functions $f_i \in \mathcal{A}_{G,S}$ attached to each $i \in I$ and a sequence of mutations taking the seed (I, I_0, B, d) to an isomorphic seed (I', I'_0, B', d') such that the functions attached to these two seeds are related by σ . More precisely, let $\phi : (I, I_0, B, d) \rightarrow (I', I'_0, B', d')$ be such an isomorphism. For any $i \in I$, let the associated function be f_i . Then we would like to show that

$$\sigma^* f_i = f_{\phi(i)}.$$

Theorem 4.1. Let $\sigma \in \text{Out}(G)$. Then σ acts on $\mathcal{A}_{G,S}$. Suppose that $\mathcal{O}(\mathcal{A}_{G,S})$ has the structure of a cluster algebra. Then there exists a seed consisting of the functions

$$f_1, f_2, \dots, f_n$$

and a sequence of mutations which transforms this seed into the seed consisting of the functions

$$\sigma^* f_1, \sigma^* f_2, \dots, \sigma^* f_n$$

in such a way that the other corresponding seed data are isomorphic.

Remark 4.2. Once one has the above theorem for one seed, the theorem for any seed is automatic by mutation.

Proof. It is enough realize σ via a sequence of mutations on each triangle.

The automorphism σ permutes the vertices of the Dynkin diagram, so that it permutes the fundamental weights. Thus we get a natural action of σ on the simple reflections $s_i \in W$. Consider the seed constructed from a reduced word for w_0 :

$$w_0 = s_{i_1} s_{i_2} \dots s_{i_K-1} s_{i_K}.$$

Because σ is an automorphism of the Dynkin diagram, there is another reduced word

$$w_0 = s_{\sigma(i_1)} s_{\sigma(i_2)} \dots s_{\sigma(i_{K-1})} s_{\sigma(i_K)}.$$

This reduced word gives a second seed.

By [BFZ] and , there is a sequence of mutations relating the seeds coming from these two reduced words. We then clearly have

$$\sigma^*(\Delta_{\sigma(u_l)\omega_{\sigma(i_l)}, \omega_{\sigma(i_l)}}) = \Delta_{u_l\omega_{i_l}, \omega_{i_l}}$$

and a similar formula holds for all the cluster functions in our two seeds. We then only need to verify that the seed data are isomorphic. But the seed data were determined by steps 1-5 in our procedure, which was completely Lie-theoretic, so that the seed data for the two seeds must be isomorphic. \square

The above theorem applies to the cases when G has type A or D_4 . We also expect it to apply for the automorphism of E_6 , dependent upon conjectures 3.1 and 3.5. In the next section, show that there is actually a seed for the cluster structure on $\text{Conf}_3 \mathcal{A}_{\text{Spin}_8, S}$ which is preserved by the outer automorphisms of Spin_8 .

One interesting interpretation of the above theorem is that the outer automorphism $\sigma \in \text{Out}(G)$ gives rise to an element of the *cluster modular group*. This group has been well-studied, for example in [FG2], [GS2], [F]. Outside of a small number of cases, the cluster modular group of $\mathcal{A}_{G, S}$ is known to include the mapping class group of the surface S . Work of Goncharov and Shen [GS2] as well as their forthcoming work shows that it also contains a copy of the Weyl group W for every hole of S without marked points. Together, the mapping class group, the copies of W , and the outer automorphisms $\sigma \in \text{Out}(G)$ give all known elements of the cluster modular group.

Let us make some remarks about when G has type A .

In [Hen], there is a sequence of mutations on $\text{Conf}_3 \mathcal{A}_{SL_N}$ realizing the outer automorphism of SL_N . We will call this sequence of mutations the *cactus sequence*. We can interpret this sequence of mutations as relating the cluster structure coming from the reduced word

$$s_1 s_2 \dots s_{N-1} s_1 s_2 \dots s_{N-2} \dots s_1 s_2 s_3 s_1 s_2 s_1$$

to the cluster structure coming from the reduced word

$$s_{N-1} s_{N-2} \dots s_1 s_{N-1} s_{N-2} \dots s_2 \dots s_{N-1} s_{N-2} s_{N-3} s_{N-1} s_{N-2} s_{N-1}.$$

5. G HAS TYPE G_2

Throughout this section, G will be the simply-connected group with type G_2 . The root system for G contains 12 roots, and there are two simple roots, which we will call α and β . We normalize so that the short root α has length 1, while the long root β has length 3. Let the corresponding simple reflections in the Weyl group be s_a and s_b .

We will consider the quiver for $\text{Conf}_3 \mathcal{A}_G$ coming from the reduced word

$$w_0 = s_b s_a s_b s_a s_b s_a.$$

This yields the following quiver for B^- :

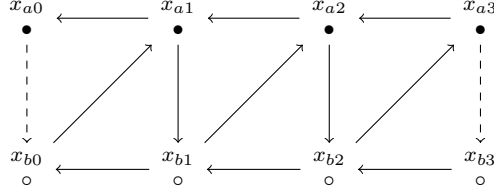


Figure 8. The quiver for the cluster algebra on B^- for G of type G_2 .

Here is how it corresponds to the reduced word for w_0 :

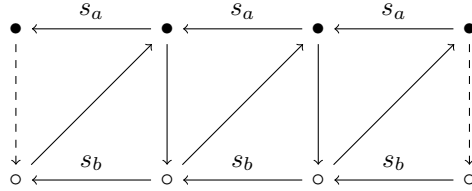


Figure 9. How the quiver for B_G^- corresponds to the reduced word $s_b s_a s_b s_a s_b s_a$.

5.1. The functions. We now need to calculate the functions attached to each vertex in the quiver. We will do this by relating the functions on B_G^- (and also $\text{Conf}_3 \mathcal{A}_G$) to those on $B_{Spin_8}^-$ (and $\text{Conf}_3 \mathcal{A}_{Spin_8}$).

Note that $Spin_8$ has a Dynkin diagram with four nodes, with the central node connected to the three others. The group S_3 therefore acts by automorphisms on the Dynkin diagram, so that $\text{Out}(Spin_8) \simeq S_3$. The group G embeds into $Spin_8$ as the fixed point set of the outer automorphism group. The embedding

$$G \xhookrightarrow{i} Spin_8$$

gives an embedding

$$\text{Conf}_3 \mathcal{A}_G \xhookrightarrow{j} \text{Conf}_3 \mathcal{A}_{Spin_8}.$$

Then all functions in the cluster algebra on $\text{Conf}_3 \mathcal{A}_G$ are pulled back from functions in the cluster algebra on $\text{Conf}_3 \mathcal{A}_{Spin_8}$. Let us elaborate upon this.

The reduced word $s_b s_a s_b s_a s_b s_a$ comes from a folding of a reduced word for the longest element of the Weyl group of $Spin_8$. The group $Spin_8$ has four simple roots. Let us call them $\alpha_1, \alpha_2, \alpha_3, \beta$, where the α_i fold to give the short root α . Let the corresponding simple reflections be $s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_3}, s_\beta$. Then a reduced word for w_0 for $Spin_8$ is given by

$$w_0 = s_\beta s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_\beta s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_\beta s_{\alpha_1} s_{\alpha_2} s_{\alpha_3}.$$

Then, using the constructions of the previous section, this reduced word gives a cluster for $\text{Conf}_3 \mathcal{A}_{Spin_8}$. The quiver for $B_{Spin_8}^-$ is depicted below.

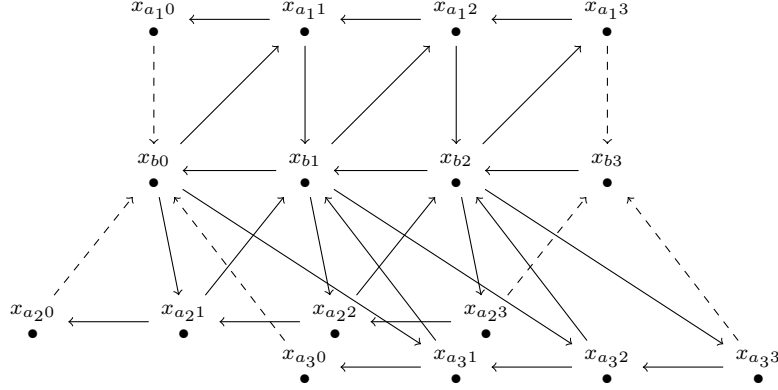


Figure 10. The cluster algebra for $B_{Spin_8}^-$.

This particular cluster is invariant under the action of $Out(Spin_8)$. It should be clear how S_3 acts on this quiver: There are vertices $x_{a_i j}$ for $1 \leq i \leq 3$ and $0 \leq j \leq 3$. The vertices $x_{a_i j}$ for fixed j are permuted among each other. The action of $\sigma \in S_3$ maps $x_{a_i j} \rightarrow x_{a_{\sigma(i)} j}$.

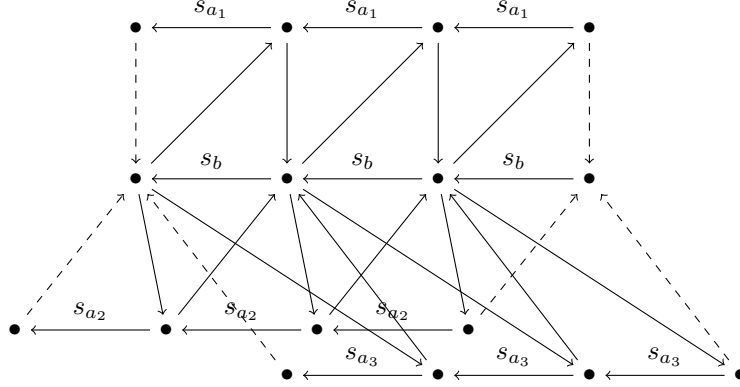


Figure 11. How the quiver for $B_{Spin_8}^-$ corresponds to the reduced word $w_0 = s_b s_{a_1} s_{a_2} s_{a_3} s_b s_{a_1} s_{a_2} s_{a_3} s_b s_{a_1} s_{a_2} s_{a_3}$.

The functions attached to the above cluster can be calculated by reduced minors. Alternatively, the above cluster is related to the one constructed in [Le] by a sequence of mutations—because both clusters come from a reduced word we can use Theorem 3.8. Therefore the functions attached to the cluster above can be written directly in terms to the functions from [Le]. Let us now explain how this allows us to construct the functions on B_G^- and $\text{Conf}_3 \mathcal{A}_G$ from those on $\text{Conf}_3 \mathcal{A}_{Spin_8}$.

Suppose that the cluster function attached to a vertex x_i is f_i . The action of $\sigma \in Out(G)$ then acts on functions by $\sigma^*(f_{a_i j}) = f_{a_{\sigma(i)} j}$. Then $B_G^- \subset B_{Spin_8}^-$ is precisely the locus of points where $f_{a_1 j} = f_{a_2 j} = f_{a_3 j}$ for all j . Moreover, we can pull back the function $f_{a_i j}$ to B_G^- to get the function $f_{a j}$.

Using the constructions of the previous section, we can complete the cluster algebra structure on $B_{Spin_8}^-$ to obtain one on $\text{Conf}_3 \mathcal{A}_{Spin_8}$. By slight abuse of notation, we will still denote by $f_{a_{ij}}$ the function $f_{a_{ij}}$ extended to $\text{Conf}_3 \mathcal{A}_{Spin_8}$. Then the functions on $\text{Conf}_3 \mathcal{A}_{Spin_8}$ will be $f_{a_{ij}}$ for $1 \leq i \leq 3$ and $0 \leq j \leq 3$ plus the functions on the third edge, which are analogous to the functions on the other two edges. Because our construction was Lie-theoretic, we still have an action of $\text{Out}(Spin_8)$ this cluster for $\text{Conf}_3 \mathcal{A}_{Spin_8}$. We can fold this cluster by the automorphisms to get the cluster with the quiver in Figure 12. In particular, the quiver contains information on how the edge vertices along the third edge attach to the cluster for B^- .

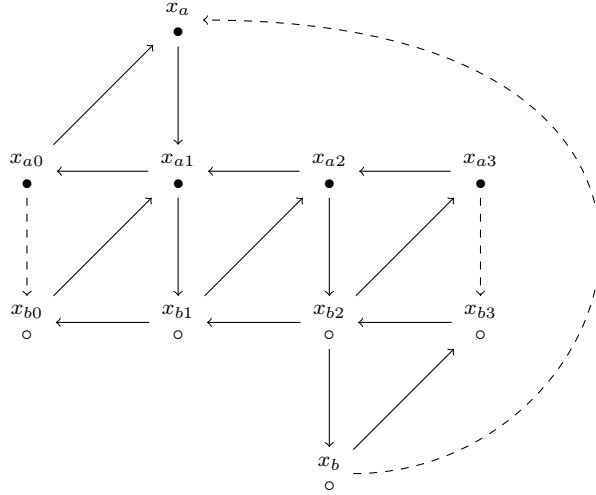


Figure 12. The quiver for the cluster algebra on $\text{Conf}_3 \mathcal{A}_G$ for G of type G_2 .

If the function f_i is attached to the vertex x_i , we will have that

$$j^*(f_{a_{ij}}) = f_{aj}.$$

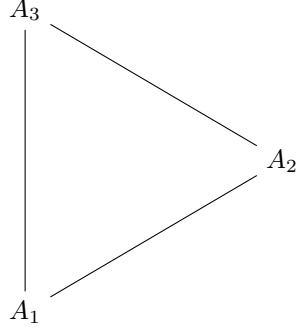
In figure 13, we put functions at the corresponding vertices. We will abbreviate by

$$\begin{pmatrix} \nu \\ \lambda \end{pmatrix} \mu$$

a function which lies in the invariant space

$$[V_\lambda \otimes V_\mu \otimes V_\nu]^G,$$

where λ, μ, ν are the gradings of the function in the space of functions on the flags A_1, A_2, A_3 respectively. We picture the three flags lying at the vertices of a triangle as follows:



For the remainder of this paper, we will not specify which particular invariant vector within $[V_\lambda \otimes V_\mu \otimes V_\nu]^G$ the function corresponds to. The particular functions can be calculated in terms of the functions in [Le]. We emphasize the weights of these functions because they are what are important for determining the quiver using the algorithm prescribed in Section 4, as well as for describing the \mathcal{X} -space. We will denote by a the fundamental weight ω_α and by b the fundamental weight ω_β .

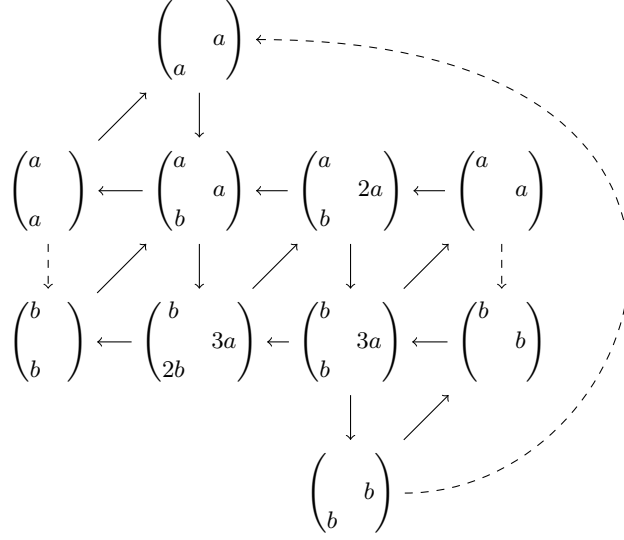
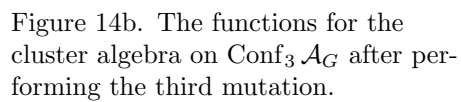
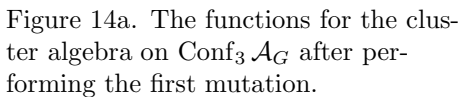


Figure 13. The functions for the cluster algebra on $\text{Conf}_3 \mathcal{A}_G$ for G of type G_2 .

5.2. The first transposition. We now exhibit the transposition associated to the transposition (13). This transposition is realized by the mutation sequence

$$(8) \quad \begin{array}{c} x_{a2}, \\ x_{a1}, x_{b1}, \\ x_{a2} \end{array}$$

We view the sequence of mutations as consisting of three stages, corresponding to the three rows in which we listed the mutations. The quiver and the functions transform as follows:



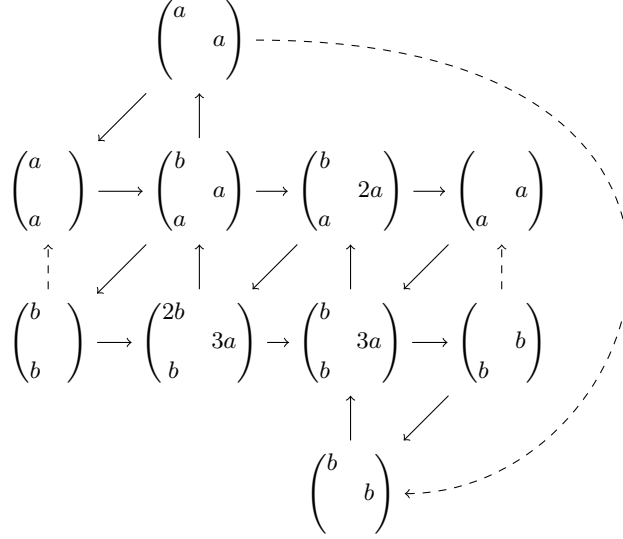


Figure 14c. The functions for the cluster algebra on $\text{Conf}_3 \mathcal{A}_G$ after performing the fourth and final mutation.

5.3. The second transposition. We now exhibit the transposition associated to the transposition (23). This transposition is realized by the mutation sequence

$$(9) \quad \begin{array}{c} x_{b1}, \\ x_{b2}, x_{a2}, \\ x_{b1} \end{array}$$

We view the sequence of mutations as consisting of three stages, corresponding to the three rows in which we listed the mutations. The quiver and the functions transform as follows:

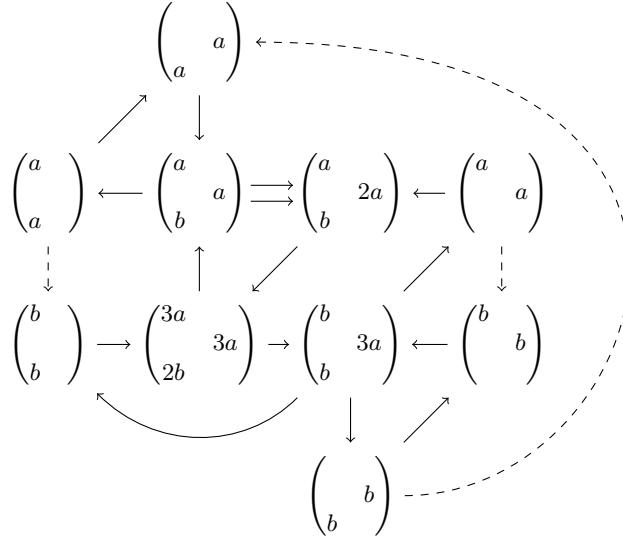


Figure 15a. The functions for the cluster algebra on $\text{Conf}_3 \mathcal{A}_G$ after performing the first mutation.

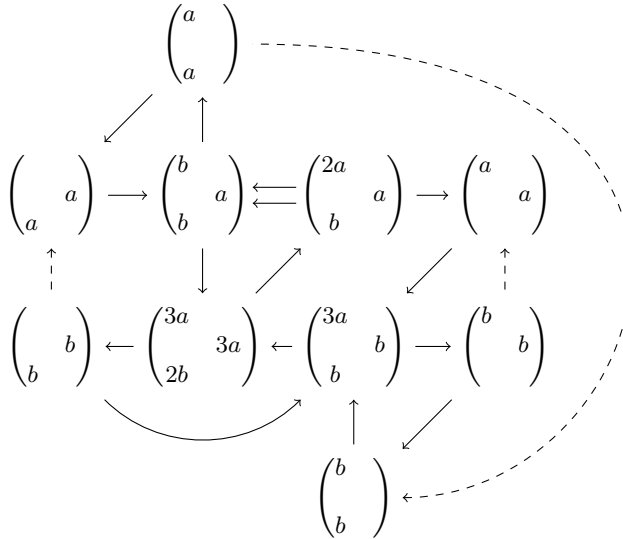


Figure 15b. The functions for the cluster algebra on $\text{Conf}_3 \mathcal{A}_G$ after performing the third mutation.

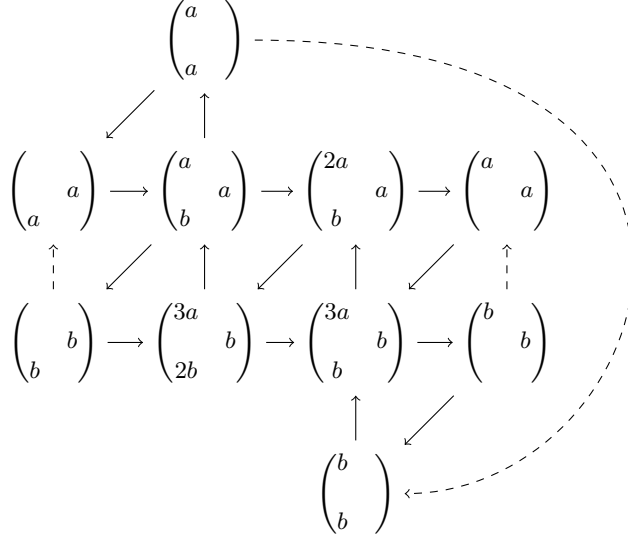


Figure 15c. The functions for the cluster algebra on $\text{Conf}_3 \mathcal{A}_G$ after performing the fourth and final mutation.

Note that the sequence of mutations associated to a transposition results in a seed isomorphic to the initial one, except with all the arrows *reversed*.

5.4. The third transposition, Langlands duality. Let us first observe that the third transposition (12) can be realized by the composition of transpositions (13)(23)(13). Thus the action of this transposition on $\text{Conf}_3 \mathcal{A}_G$ can be realized via the sequence of mutations

$$(10) \quad \begin{aligned} & x_{a2}, x_{a1}, x_{b1}, x_{a2} \\ & x_{b1}, x_{b2}, x_{a2}, x_{b1} \\ & x_{a2}, x_{a1}, x_{b1}, x_{a2} \end{aligned}$$

On the other hand, recall that the the third transposition relates the clusters attached to the reduced words $s_b s_a s_b s_a s_b s_a$ and $s_a s_b s_a s_b s_a s_b$. Thus it can be realized as a sequence of mutations using the work of [BFZ], and [BZ], where this sequence first appeared. Thus we have given an alternative proof of the following theorem:

Theorem 5.1. [BZ] There exists a sequence of mutations relating the reduced words $s_b s_a s_b s_a s_b s_a$ and $s_a s_b s_a s_b s_a s_b$.

The same result was later rederived in [FG3]. In all both [BZ] and [FG3], the computation of the sequence comes down to an involved, and not very illuminating, computation. Our analysis, then, gives a more conceptual proof of the existence of such a sequence of mutations: we relate the sequence to the transposition (12) and break down the sequence of mutations into three stages corresponding to the decomposition $(12) = (13)(23)(13)$.

This suggests, then, that the cluster algebra for $\text{Conf}_3 \mathcal{A}_G$ is in some sense *more fundamental* than the one for B_G^- . Note that the transpositions (13) and (23) take us outside the world of reduced word decompositions in order to prove a fundamental fact about relating reduced word decompositions.

An analogous result is also true in the case of where G has type $B_2 = C_2$, i.e., when our group is $Spin_5 = Sp_4$. As explained in [Le], the transpositions (13) and (23) for the group Sp_4 each consist of one mutation. They compose to give the transposition (12) = (13)(23)(13) in a sequence of three mutations. This sequence of mutations relates the reduced words $s_1 s_2 s_1 s_2$ and $s_2 s_1 s_2 s_1$.

Finally, we wish to say some words about Langlands duality. The group G_2 is self-dual. The duality interchanges the simple roots α and β . This manifests itself in a symmetry for the quiver we have constructed for $\text{Conf}_3 \mathcal{A}_G$: if we rotate the quiver by 180 degrees, switch the coloring of all the vertices, and reverse all the arrows, we obtain the same quiver again.

Moreover, this symmetry is stable under mutation. Let us be more precise. Let L be the transformation of the quiver which reverses all arrows, switches the coloring of all vertices, and interchanges the following pairs of vertices of the quiver:

$$\begin{aligned} x_a &\longleftrightarrow x_b \\ x_{a0} &\longleftrightarrow x_{b3} \\ x_{a1} &\longleftrightarrow x_{b2} \\ x_{a2} &\longleftrightarrow x_{b1} \\ x_{a3} &\longleftrightarrow x_{b0} \end{aligned}$$

Then consider any sequence of mutations, x_{i_1}, \dots, x_{i_k} . Then mutating x_{i_1}, \dots, x_{i_k} and then applying L gives the same result as applying L and then mutating $L(x_{i_1}), \dots, L(x_{i_k})$.

The reason this happens is that the cluster algebra on $\text{Conf}_3 \mathcal{A}_G$ is Langlands dual to itself:

Definition 5.2. [FG2] Two seeds that have the same set of vertices I and the same set of frozen vertices I_0 are said to be *Langlands dual* if they have B -matrices b_{ij} and b_{ij}^\vee and multipliers d_i and d_i^\vee where

$$\begin{aligned} d_i &= (d_i^\vee)^{-1} D, \\ b_{ij} d_j &= -b_{ij}^\vee d_j^\vee, \end{aligned}$$

for some rational number D .

Remark 5.3. Note that the multipliers d_i for a cluster algebra are determined only up to simultaneous scaling by a rational number. Conventions sometimes differ on how to specify the values for the d_i . This is one reason the rational number D appears in the above definition.

In other words, Langlands duality on seeds involves switching colors of vertices and reversing all arrows. This procedure applied to $\text{Conf}_3 \mathcal{A}_G$ gives the same seed with the vertices rearranged. Amalgamation gives the same result for $\text{Conf}_m \mathcal{A}_G$.

Up until now, we have only discussed Langlands duality on the level of seeds. Let us discuss what it does on the level of functions, or cluster variables.

When two seeds are Langlands dual, there is a close relationship between the resulting cluster algebras. Suppose that (I, I_0, b_{ij}, d_i) and $(I, I_0, b_{ij}^\vee, d_i^\vee)$ are Langlands dual seeds. Let the cluster variables for the initial seeds be f_1, \dots, f_n and $f_1^\vee, \dots, f_n^\vee$, respectively. These cluster variables are naturally in bijection. Then if we mutate f_k to obtain the new cluster variable f'_k , we can do the same to f_k^\vee to get $(f'_k)^\vee$ and then match f'_k and $(f'_k)^\vee$. Continuing in this manner, one conjecturally gets a bijection between all the cluster variables for the Langlands dual seeds.

We now explain the relationship between the functions f'_k and $(f'_k)^\vee$. We have an isomorphism between the weight spaces and the coweight spaces for the group G given by the Killing form, which takes

$$\begin{aligned} \alpha &\rightarrow \alpha^\vee \\ \beta &\rightarrow 3\beta^\vee \end{aligned}$$

On the other hand, the isomorphism between G and its Langlands dual means that $\alpha^\vee = \beta$ and $\beta^\vee = \alpha$. Composing these gives an isomorphism L from the weight space for G to itself which maps (here we describe the map on fundamental weights instead of simple roots)

$$a \rightarrow b$$

$$b \rightarrow 3a$$

We have the following observation:

Observation 5.4. Suppose that we have a cluster variable

$$f \in [V_\lambda \otimes V_\mu \otimes V_\nu]^G \subset \mathcal{O}(\text{Conf}_3 \mathcal{A}_G).$$

Then if f is associated to a black vertex and f^\vee is associated to a white vertex, then

$$f^\vee \in [V_{L(\lambda)} \otimes V_{L(\mu)} \otimes V_{L(\nu)}]^G \subset \mathcal{O}(\text{Conf}_3 \mathcal{A}_G)$$

while if f is associated to a white vertex and f^\vee is associated to a black vertex

$$f^\vee \in [V_{L(\lambda)/3} \otimes V_{L(\mu)/3} \otimes V_{L(\nu)/3}]^G \subset \mathcal{O}(\text{Conf}_3 \mathcal{A}_G).$$

This is clearly true in the initial cluster, and as long as all the cluster variables are functions on $\text{Conf}_3 \mathcal{A}_G$ and therefore given by tensor invariants, and not just rational functions on $\text{Conf}_3 \mathcal{A}_G$ (as we expect, but do not know how to prove), it is easy to check that the above observation remains true under mutation. Certainly, in all the clusters we consider in this paper this will be the case. For example, the Langlands dual to the seed in Figure 13 is given by applying the transposition (12) to the original seed for $\text{Conf}_3 \mathcal{A}_G$, giving us a third way of looking at this transposition!

One last observation:

Observation 5.5. The sequence of mutations for the transpositions (13) and (23) are related by Langlands duality. Moreover, the sequence of mutations of the flip (discussed in the next section) and the reverse of the sequence of mutations for a flip are related by Langlands duality. An analogous statement is true for the group Sp_4 .

5.5. The sequence of mutations for a flip. In this section, we will give a sequence of mutations that relates two of the clusters coming from different triangulations of the 4-gon. Combined with the previous section, this allows us to connect by mutations all 72 different clusters we have constructed for $\text{Conf}_4 \mathcal{A}_G$.

Given a configuration $(A, B, C, D) \in \text{Conf}_4 \mathcal{A}_G$, we will give a sequence of mutations that relates a cluster coming from the triangulation ABC, ACD to a cluster coming from the triangulation ABD, BCD . Refer to Figure 6 for a diagram of the configuration (A, B, C, D) .

We will need to relabel the quiver with vertices x_{ij} , y_k , with $j = a, b$, $-n \leq i \leq n$, and $k = \pm a, \pm b$. The quiver we will start with is as in Figure 16.

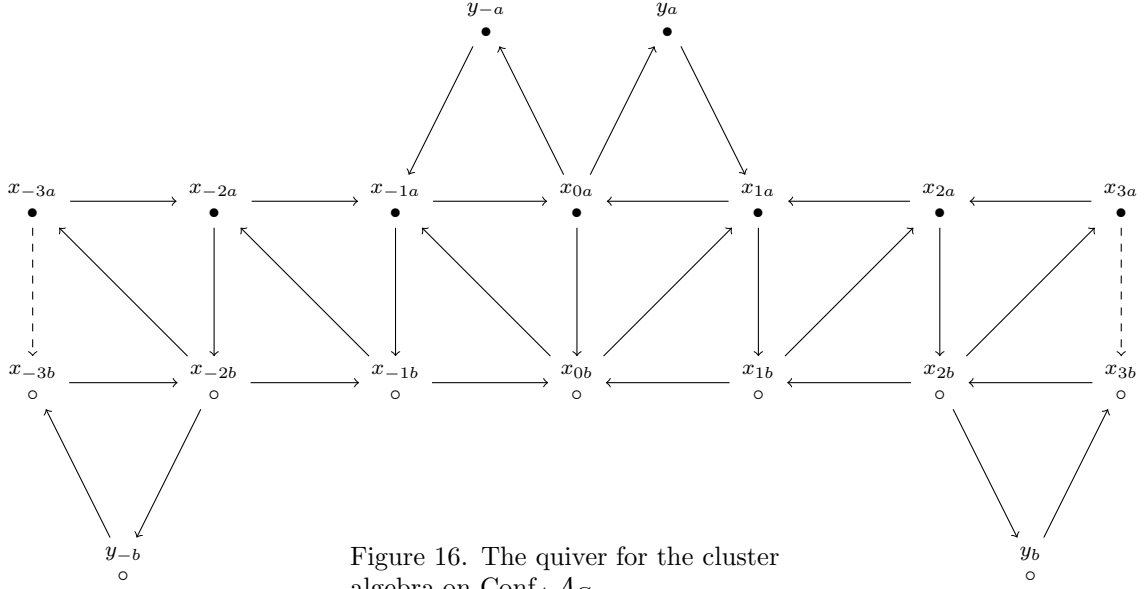


Figure 16. The quiver for the cluster algebra on $\text{Conf}_4 \mathcal{A}_G$.

We have removed the dashed arrows between y_b and y_a and between y_{-b} and y_{-a} for simplicity. These arrows neither change nor play any role during our sequence of mutations. We write down the functions associated to the vertices in the quiver in Figure 17.

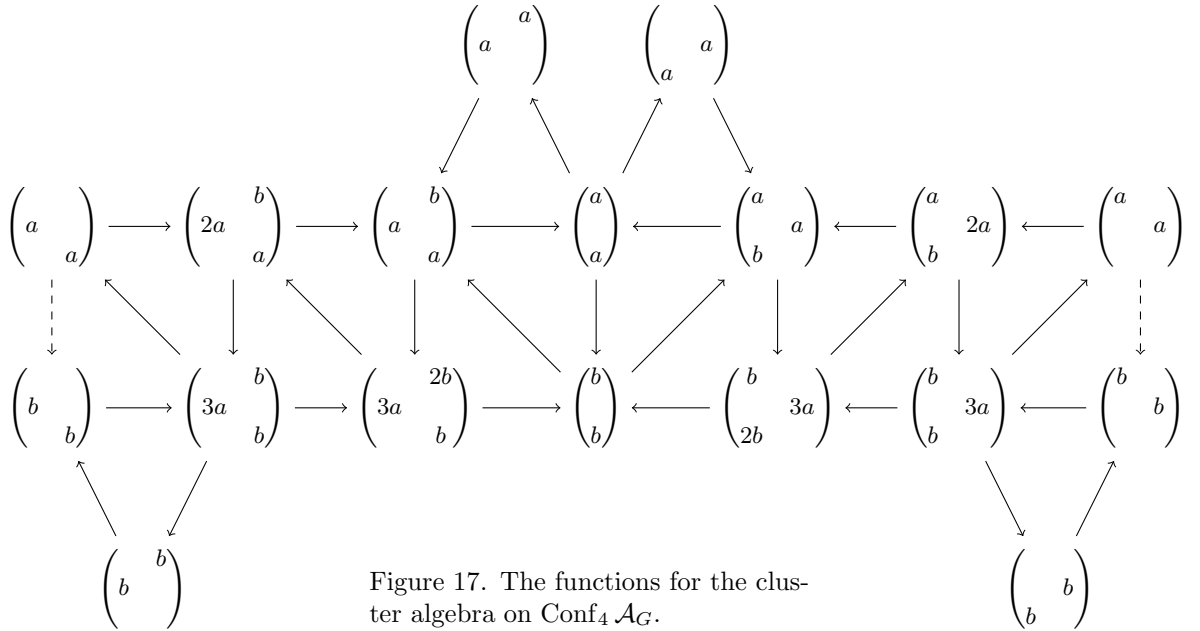


Figure 17. The functions for the cluster algebra on $\text{Conf}_4 \mathcal{A}_G$.

The sequence of mutations for the flip of a triangulation is as follows:

$$\begin{aligned}
(11) \quad & x_{0a}, \\
& x_{-1a}, x_{0b}, x_{1a}, \\
& x_{-2a}, x_{-1b}, x_{0a}, x_{1b}, x_{2a}, \\
& x_{-2b}, x_{-1a}, x_{0b}, x_{1a}, x_{2b}, \\
& x_{-1b}, x_{0a}, x_{1b}, \\
& x_{0b},
\end{aligned}$$

The rows above correspond to what we will call the six stages of the mutation sequence. In each stage, the vertices may be mutated in any order. We depict how the quiver and the functions changes after each stage of the sequence of mutations in Figure 18.

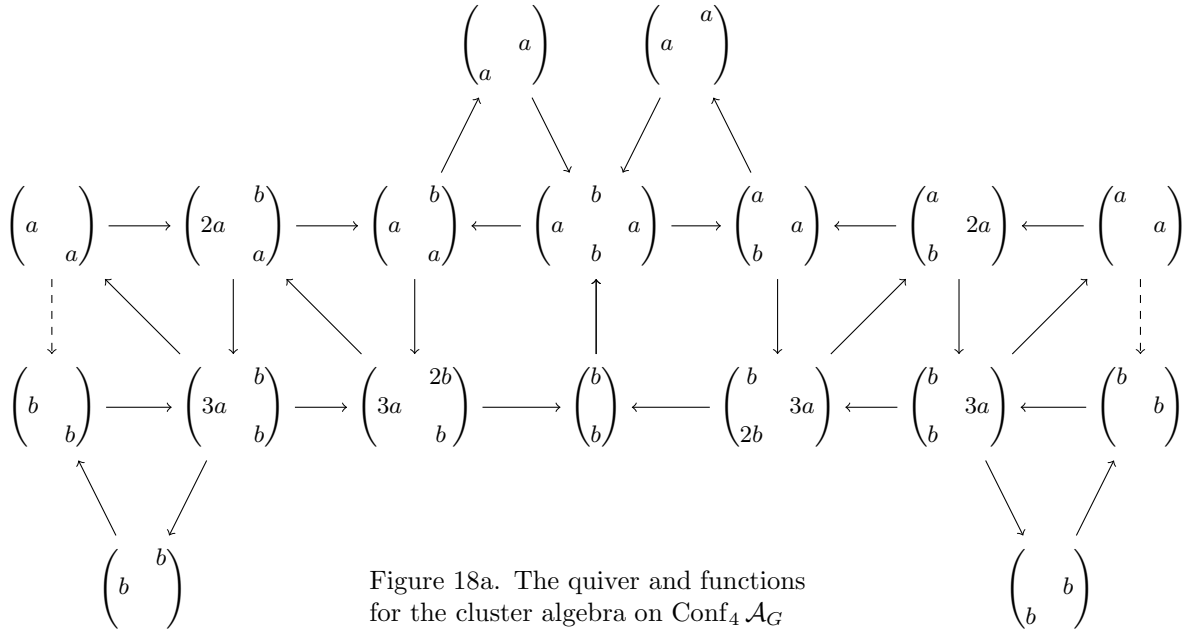


Figure 18a. The quiver and functions for the cluster algebra on $\text{Conf}_4 \mathcal{A}_G$ after the first stage of mutation.

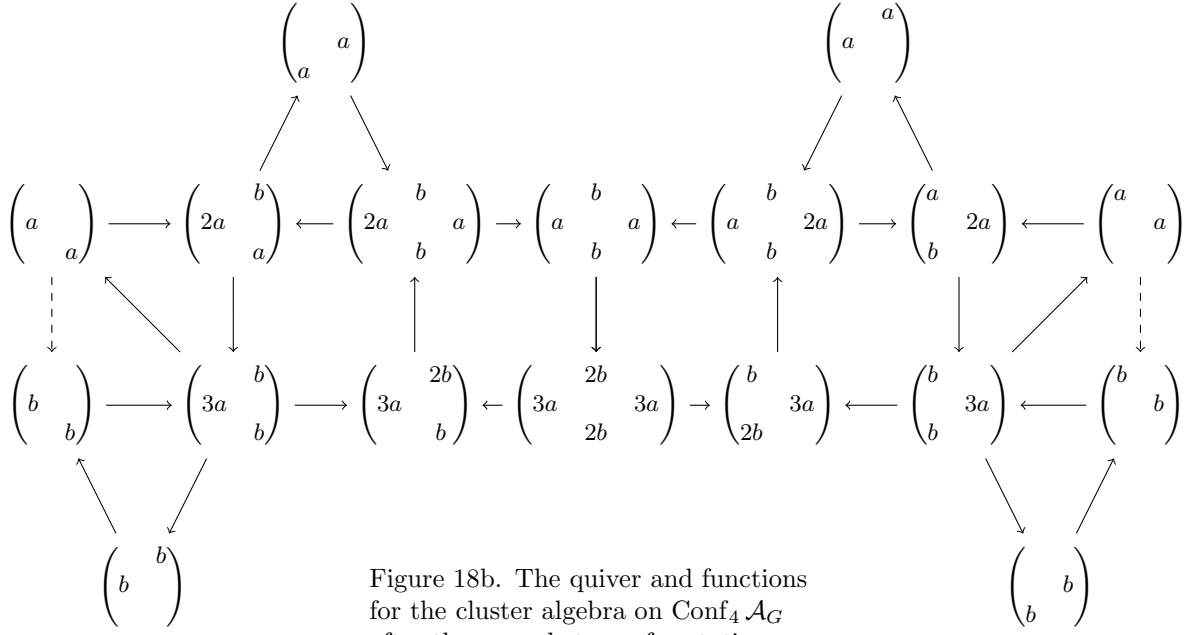


Figure 18b. The quiver and functions for the cluster algebra on $\text{Conf}_4 \mathcal{A}_G$ after the second stage of mutation.

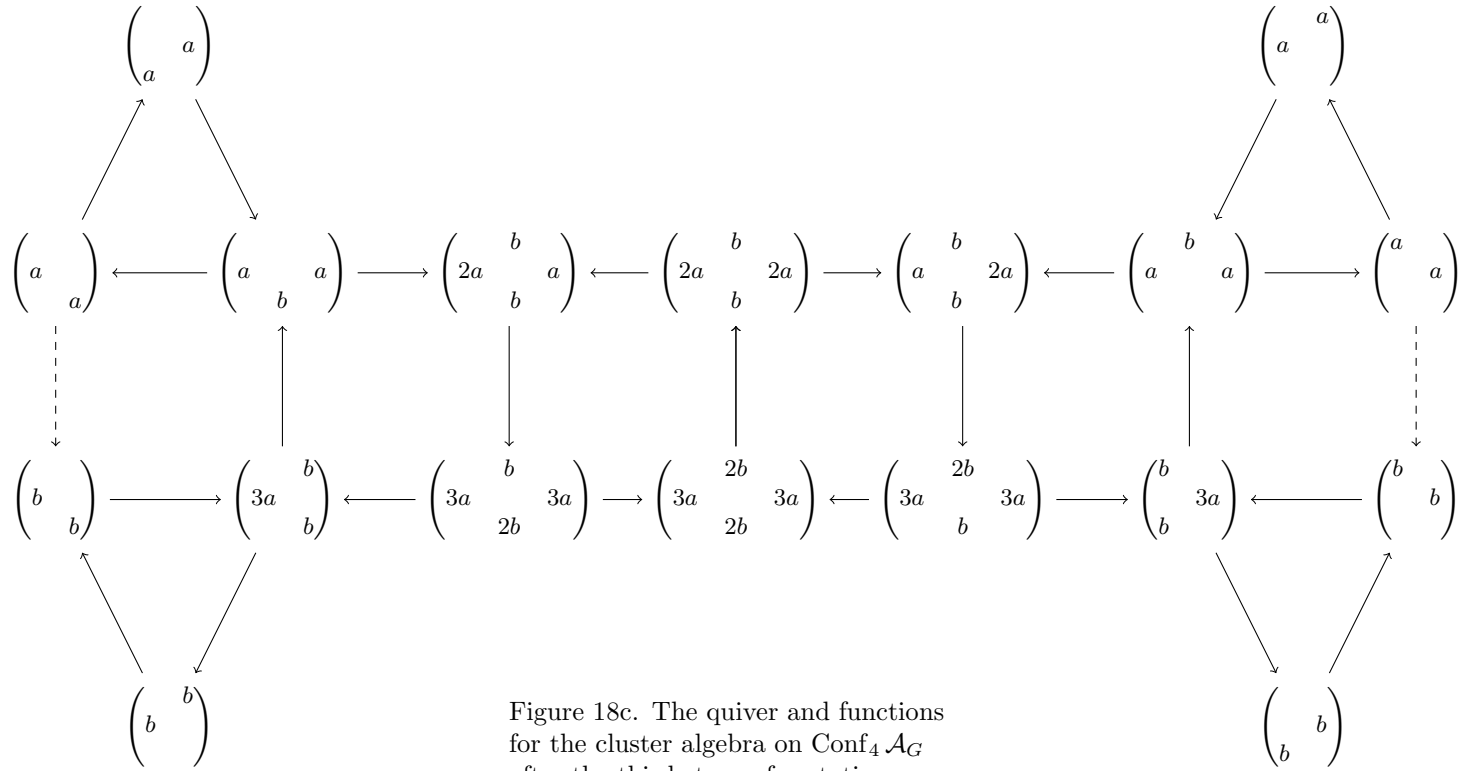
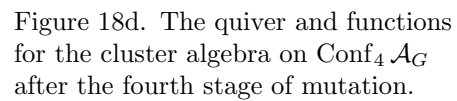


Figure 18c. The quiver and functions for the cluster algebra on $\text{Conf}_4 \mathcal{A}_G$ after the third stage of mutation.



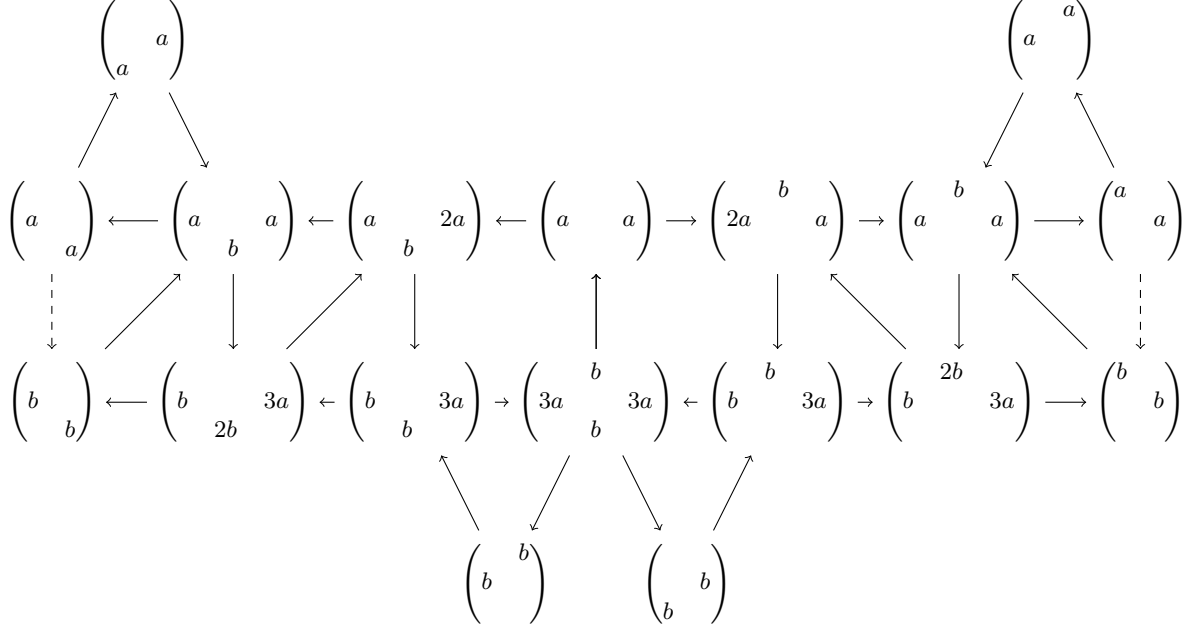


Figure 18e. The quiver and functions for the cluster algebra on $\text{Conf}_4 \mathcal{A}_G$ after the fifth stage of mutation.

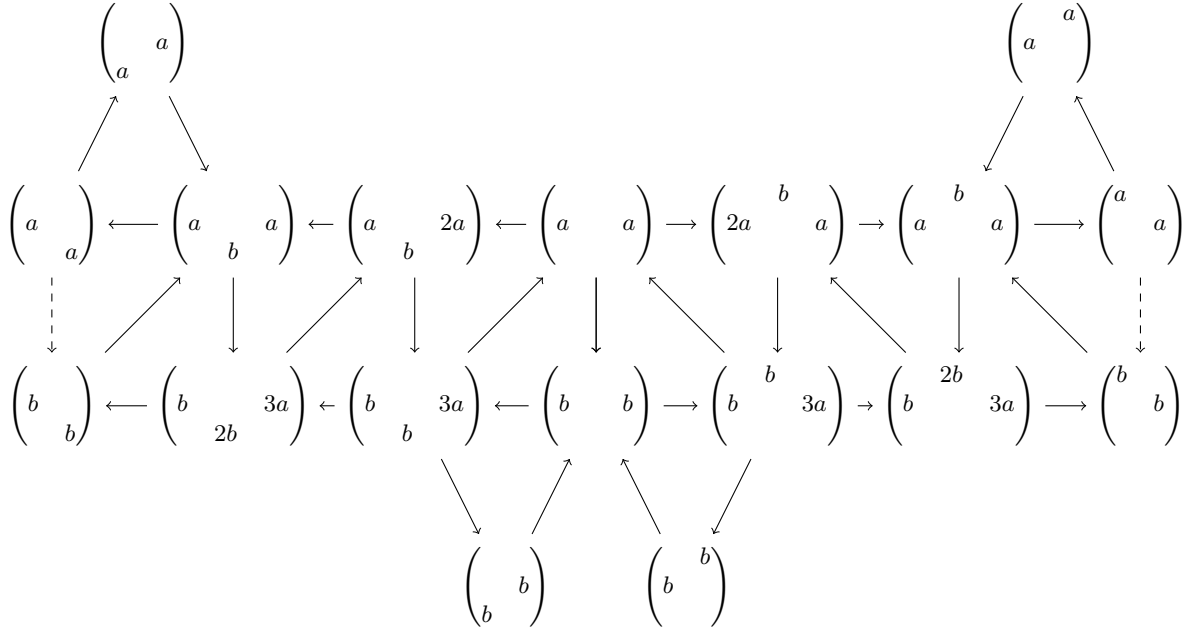


Figure 18f. The quiver and functions for the cluster algebra on $\text{Conf}_4 \mathcal{A}_G$ after the sixth and final stage of mutation.

5.6. The space $\mathcal{X}_{G,S}$. Let us mention here that once we have constructed the cluster algebra structure on the spaces $\text{Conf}_m \mathcal{A}_G$, it is straightforward to derive the \mathcal{X} -variety structure on $\text{Conf}_m \mathcal{B}_G$. The general framework is explained in [FG2], and some details specific to the case of configurations of principal flags and configurations of flags can be found in [Le]. We give a short summary here which is adapted to the general case. The statements below are conditional upon the conjectures from section 3, and thus hold for types A, B, C, D and G_2 .

Theorem 5.6. $\text{Conf}_m \mathcal{B}_G$ has the structure of a cluster \mathcal{X} -variety. This is the \mathcal{X} -variety which is attached, via the constructions of [FG2], to cluster structure that we have constructed on $\text{Conf}_m \mathcal{A}_G$.

Suppose we have a cluster \mathcal{A} -variety with seed $\Sigma = (I, I_0, B, d)$. Then for every non-frozen index $i \in I$, there is a cluster variable X_i . There is a map from $p : \mathcal{A}_\Sigma \rightarrow \mathcal{X}_\Sigma$ given by

$$p^*(X_i) = \prod_{j \in I} A_j^{B_{ij}}.$$

The functions $p^*(X_i)$, which a priori live on $\text{Conf}_3 \mathcal{A}_G$, turn out to descend to $\text{Conf}_3 \mathcal{B}_G$. The reason is that the cluster functions A_j that we constructed on $\text{Conf}_3 \mathcal{A}_G$ were invariants of tensor products:

$$A_j \in [V_\lambda \otimes V_\mu \otimes V_\nu]^G.$$

Now recall that G/U has a left action of H , the Cartan subgroup. The functions on G/U decompose as

$$\bigoplus_{\lambda \in \Lambda_+} V_\lambda.$$

Moreover, $h \in H$ acts on the summand V_λ by $\lambda(h)$.

Correspondingly, on $\text{Conf}_3 \mathcal{A}_G$ there is an action of H^3 , and (h_1, h_2, h_3) acts on the summand

$$[V_\lambda \otimes V_\mu \otimes V_\nu]^G$$

by

$$\lambda(h_1)\mu(h_2)\nu(h_3).$$

Recall that the quiver for our cluster algebra was chosen precisely so that

$$(12) \quad \sum_{j \in I} B_{ij}(\lambda_j, \mu_j, \nu_j) = 0$$

This forces the action of H^3 on the functions X_i to be trivial.

Now we must check that the torus \mathcal{X}_Σ is birational to $\text{Conf}_3 \mathcal{B}_G$. From the above, we have a map $p' : \text{Conf}_3 \mathcal{B}_G \rightarrow \mathcal{X}_\Sigma$, because the functions X_i can be viewed as functions on $\text{Conf}_3 \mathcal{B}_G$. We will show that they parameterize an open set in $\text{Conf}_3 \mathcal{B}_G$.

To do this, we will adapt the results of [FG3] and [W] on parameterization of double Bruhat cells $G^{u,v}$, applied to the particular Bruhat cell $G^{w_0,e}$. Let us recall the setup. Recall that the functions on $\text{Conf}_3 \mathcal{A}_G$ were associated to a reduced word composition for w_0 . Suppose the Weyl group for G is generated by n simple reflections s_i , $1 \leq i \leq n$. Take a reduced word

$$w_0 = s_{i_1} s_{i_2} s_{i_3} \cdots s_{i_{K-1}} s_{i_K}$$

Recall that if the simple reflection s_i occurs a_i times, then there are $a_i + 1$ cluster variables attached to the simple reflection s_i . The first and last of these are frozen, and do not correspond to \mathcal{X} -variables. Thus we can naturally put the \mathcal{X} -variables in bijection with the simple reflections s_{i_j} except those s_{i_j} which are the leftmost occurrence of some simple reflection. For

simplicity, let us suppose that s_{i_1}, \dots, s_{i_n} are s_1, \dots, s_n in some order. Then we may put the X variables in bijection with s_{i_j} for $n < j \leq K$. Let X_j be the \mathcal{X} -function attached to the simple reflection s_{i_j} for $n < j \leq K$.

It is known that there is a parameterization of $\text{Conf}_3 \mathcal{B}_G$ given by three flags

$$(B^+, u^- B^+, B^-),$$

where u^- is determined up to the adjoint action of H . Let b^- be an element of B^- . Then there is a natural projection

$$\pi : B^- \rightarrow H = B/[B, B]$$

The choice of opposite flags B^+ and B^- gives an inclusion

$$i : H \rightarrow B^-.$$

Then let

$$\rho(b^-) := i(\pi(b^-))^{-1}b^-.$$

This associates to each element of B^- an element of U^- . We will be interested in $\rho(b^-)$ up to the adjoint action of H .

Then the co-ordinates X_j give a parameterization of u^- by the following formula:

$$u^- = \rho(b^-),$$

where

$$\begin{aligned} b^- &= \left(\prod_{j=n+1}^K F_{i_j} H_{\omega_{i_j}^\vee}(X_j^{-1}) \right) F_n \dots F_3 F_2 F_1 \\ &= F_{i_{n+1}} H_{\omega_{i_{n+1}}^\vee}(X_{n+1}^{-1}) F_{i_{n+2}} H_{\omega_{i_{n+2}}^\vee}(X_{n+2}^{-1}) \dots \\ &\quad F_{i_{K-1}} H_{\omega_{i_{K-1}}^\vee}(X_{K-1}^{-1}) F_{i_K} H_{\omega_{i_K}^\vee}(X_K^{-1}) F_n \dots F_3 F_2 F_1. \end{aligned}$$

Let us explain the notation above. Here, F_i are the usual generators of the u^- associated to the simple roots. ω_i^\vee is the fundamental weight attached to the i^{th} node of the Dynkin diagram for the simply connected form of G^\vee , the Langlands dual of G . (Unlike before, ω_i^\vee is not the fundamental weight $-w_0(\omega_i)$.) The weights for G^\vee are the coweights for the adjoint form of G , and $H_{\omega_i^\vee}$ is the cocharacter attached to this coweight.

Thus the functions X_j give a parameterization of $\text{Conf}_3 \mathcal{B}_G$.

We then need to check that for any gluing of triangles to get a structure of an \mathcal{A} -space on $\text{Conf}_4 \mathcal{A}_G$, the \mathcal{X} -coordinates on the edge gluing the two triangles parameterize gluings of configurations in $\text{Conf}_3 \mathcal{B}_G$ to get a configuration in $\text{Conf}_4 \mathcal{B}_G$. In other words, there is an equivalence

$$\text{Conf}_4 \mathcal{B}_G \simeq \text{Conf}_3 \mathcal{B}_G \times H \times \text{Conf}_3 \mathcal{B}_G.$$

Let us examine the \mathcal{X} -coordinates on the \mathcal{X} -space corresponding to the \mathcal{A} -space $\text{Conf}_4 \mathcal{A}_G$. We showed above that the face \mathcal{X} -coordinates parameterize the two copies of $\text{Conf}_3 \mathcal{B}_G$. Then we need to see that the edge \mathcal{X} coordinates parameterize H , the space of gluings.

Explicitly, we have that H acts by shearing the configuration of four flags in the following way:

$$h : (B^+, u^- B^+, B^-, u^+ B^-) \rightarrow (B^+, u^- B^+, B^-, h u^+ B^-).$$

It is then a simple matter to check that the edge \mathcal{X} -coordinates a torsor for H . Suppose that the function A_j is the edge function on the glued edge given by an invariant in the space

$$[V_{\omega_j} \otimes V_{\omega_j^\vee}]^G.$$

Then let the corresponding \mathcal{X} -coordinate be X_j .

Proposition 5.7. An element $h \in H$ acts on X_j by $\alpha_j(h)$, where α_j is the j -th simple root of G' .

Note that because G' is adjoint, the simple roots α_j span the weight lattice.

Proof. The proof reduces to a computation. A similar theorem was proved for G of type A, B, C, D in [Le]. The crux of the computation is Conjecture 3.6, which can be verified by hand for the group of type G_2 using the identification of the weights of the cluster functions given in the previous section. □

Thus the edge coordinates on the \mathcal{X} space give the usual “shear” coordinates. The usual cutting and gluing arguments allow us to conclude the following:

Theorem 5.8. The spaces $\mathcal{A}_{G,S}$ and $\mathcal{X}_{G',S}$ together have the structure of a cluster ensemble when G has type G_2 .

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